A NOTE ON A THEOREM OF GANEA, HILTON AND PETERSON

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Introduction. Let $X$ be a space. We are interested in the question whether or not the loop space $\Omega X$ and the suspension $\Sigma X$ are homotopy commutative, that is whether or not $\text{nil } X \leq 1$, $\text{conil } X \leq 1$ respectively. Let $i: X \hookrightarrow X \vee X$ be the fibre of the inclusion $j: X \vee X \to X \times X$. Let $\nabla: X \vee X \to X$ be the folding map. Then in [3], Ganea, Hilton and Peterson proved the following

Theorem 1. Let $X$ be 1-connected. Then $\text{nil } X \leq 1$ if and only if

$$\nabla i = 0.
$$

Dually, let $q: X \times X \to X \wedge X$ be the cofibre of the inclusion $j$, and let $\Delta: X \to X \times X$ be the diagonal map. Let $e': X \wedge X \to \Omega(\Sigma(X \wedge X))$ be the canonical imbedding. Then in [3], the authors also proved

Theorem 2. Let $X$ be 0-connected. Then $\text{conil } X \leq 1$ if and only if

$$e'q\Delta = 0.
$$

This paper represents an attempt to understand these theorems. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \to \Omega(X \vee X)$ be the commutator map. We shall define below a map $c: \Sigma(X \times X) \to X \vee X$ obtained from $c$. Applying the co-Hopf construction, we have a map $H(c): \Sigma(\Omega X \times \Omega X) \to \Omega(X \bowtie X)$. Then we prove

Theorem 3. $c = \Omega(\nabla i)H(c)e': \Omega X \times \Omega X \to \Omega X$, the commutator map.

We observe, of course, that the condition for nil $X \leq 1$ is precisely $c = 0$. Dually, let $c': \Sigma(X \times X) \to \Sigma(X \times X) \vee \Sigma(X \times X)$ be the cocommutator map. This gives a map $c': X \times X \to \Omega(\Sigma X \vee \Sigma X)$. The Hopf construction then gives a map $J(c'): \Sigma(X \wedge X) \to \Sigma(\Sigma X \vee \Sigma X)$. Let $e: \Sigma(\Sigma X \vee \Sigma X) \to \Sigma X \vee \Sigma X$ be the map having $1_{\Sigma(X \vee X \Sigma X)}$ as its adjoint. Let us denote the cocommutator product $\Sigma X \to \Sigma X \vee \Sigma X$ by $c'$ also. The condition for conil $X \leq 1$ is precisely $c' = 0$. We prove

Theorem 4. $c' = eJ(c')\Sigma(q\Delta): \Sigma X \to \Sigma X \vee \Sigma X$.

We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.
Let $A, B$ be spaces. We have the fibration $A \rightharpoonup B \hookrightarrow A \vee B$. We can find a map $\chi: \Omega(A \times B) \to \Omega(A \vee B)$ such that $(\Omega j)\chi \simeq 1_{\Omega(A \vee B)}$. In fact we can take $\chi = \Omega(i_A p_A) + \Omega(i_B p_B)$ where $p_A, p_B$ are the projections of $A \times B$ onto the factors and $i_A: A \to A \vee B$, $i_B: B \to A \vee B$ are the inclusions. The exact sequence of the fibration now shows that there exists a unique element $[g] \in \Omega(A \vee B)$, $\Omega(A \rightharpoonup B)$ such that $1_{\Omega(A \vee B)} = (\Omega j)g + \chi(\Omega j)$.

Now for any space $X$ and a map $f: X \to A \vee B$ we can form the map $H(f) = g(\Omega f): \Omega X \to \Omega(A \rightharpoonup B)$. We shall call this the co-Hopf construction. The element $[H(f)]$ is the unique element of $[\Omega X, \Omega(A \rightharpoonup B)]$ satisfying $[\Omega f] = (\Omega i)_* [H(f)] + [\chi(\Omega f)] = (\Omega i)_* [H(f)] + [\Omega (i_A \pi_A f)] + [\Omega (i_B \pi_B f)]$ where $\pi_A: A \vee B \to A$, $\pi_B: A \vee B \to B$ are induced by the projections onto the factors.

For spaces $X$, $Y$ we have a bijection $\tau: \{[X, Y] \to [X, \Omega Y]\}$ which takes each map to its adjoint. Suppose $X$ is a given space. We have a projection $p: \Sigma \Omega X \to X$ such that $\tau(p) = 1_{\Omega X}$. Let $p_1 = i_1 p$, $p_2 = i_2 p$ where $i_1$, $i_2: X \to X \vee X$ are the injections in the first and second copies of $X$ respectively. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \to \Omega(X \vee X)$ be the commutator map. Then we can form the map $\bar{\epsilon} = \tau^{-1}\{\epsilon(\tau(p_1) \times \tau(p_2))\}: \Sigma(\Omega X \times \Omega X) \to \Omega X \vee X$. It is now easily verified that $\nabla \bar{\epsilon} = \tau^{-1}(\epsilon)$. The co-Hopf construction, applied to $\bar{\epsilon}$, gives an element $H(\bar{\epsilon})\Sigma(\Omega X \times \Omega X) \to \Omega(X \rightharpoonup X)$. Let $e': \Omega X \times \Omega X \to \Omega \Sigma(\Omega X \times \Omega X)$ be such that $e' = \tau(1_{\Omega X \times \Omega X})$. It is easily seen that $\Omega(\tau^{-1}(\epsilon))e' = c: \Omega X \times \Omega X \to \Omega X$, the commutator map. Since $\nabla \bar{\epsilon} = \tau^{-1}(\epsilon)$, Theorem 3 follows immediately from

**Theorem 5.** $\Omega(\nabla \bar{\epsilon}) = \Omega(\nabla i)H(\bar{\epsilon})\Sigma(\Omega X \times \Omega X) \to \Omega X$.

**Proof.** $H(\bar{\epsilon})$ satisfies $\Omega \bar{\epsilon} = (\Omega i)_* H(\bar{\epsilon}) + \Omega(i_1 \pi_1 \bar{\epsilon}) + \Omega(i_2 \pi_2 \bar{\epsilon})$ where $\pi_1, \pi_2: X \vee X \to X$ are induced by the projections onto the factors, and $i_1, i_2: X \to X \vee X$ are the imbeddings in the first and second copies of $X$ respectively. We have $\Omega(\nabla \bar{\epsilon}) = \Omega(\nabla i)H(\bar{\epsilon}) + \Omega(\nabla i_1 \pi_1 \bar{\epsilon}) + \Omega(\nabla i_2 \pi_2 \bar{\epsilon})$. Let $\phi$ be the loop multiplication on $\Omega X$ and $\mu$ the loop inverse. Then a simple check shows that $\tau(\nabla i_1 \pi_1 \bar{\epsilon}) = \phi(1 X) \Delta \times \phi(1 X) \Delta \mu \Delta r_1$ where $\Delta$ is the diagonal map and $r_1: \Omega X \times \Omega X \to \Omega X$ is the projection onto the first factor. Since $\phi(1 X) \Delta \simeq 1$ and $\phi(1 X) \Delta \simeq \ast$, we have $\tau(\nabla i_1 \pi_1 \bar{\epsilon}) = 0$. Hence $\nabla i_1 \pi_1 \bar{\epsilon} = 0$. Similarly $\nabla i_2 \pi_2 \bar{\epsilon} = 0$. It follows then that $\Omega(\nabla \bar{\epsilon}) = \Omega(\nabla i)H(\bar{\epsilon})$.

Theorems 1 and 3 are the immediate

**Corollary.** Let $X$ be 1-connected. If $\Omega(\nabla i) = 0$, then $\nabla i = 0$.

**Remark.** In [3], it is shown that there exist maps $a$, $b$ such that $ba = 1$, $ib = \bar{\epsilon}$. It is clear from the above that $H(\bar{\epsilon}) = \Omega b$. 

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2. We now dualise. Let \( p_1, p_2: X \times X \to X \) be the projections, and let \( e_i = e'_i p_i \) where \( e': X \to \Omega(\Sigma X) \) is the canonical imbedding. Let \( c' \) be the cocommutator map \( \Sigma(X \times X) \to \Sigma(X \times X) \setminus \Sigma(X \times X) \). Let \( c' = \tau \{ (\tau^{-1}(e_1) \lor \tau^{-1}(e_2))c' \} : X \times X \to \Omega(\Sigma X \setminus \Sigma X) \). Then \( c' \triangle = \tau(c') \) where \( \triangle \) is the diagonal map.

Let \( A, B \) be spaces. We consider the cofibration \( A \sqcup B \to A \times B \). There exists a map \( p: \Sigma(A \times B) \to \Sigma(A \sqcup B) \) such that \( p(\Sigma j) \simeq 1_{\Sigma(A \times B)} \). The exact sequence of the cofibration now shows that \( (\Sigma q)^{\#} \) is a monomorphism. Dual to the above, we now see that there exists a unique element \( [d] \in [\Sigma(A \sqcup B), \Sigma(A \times B)] \) satisfying \( 1_{\Sigma(A \times B)} = d(\Sigma q) + (\Sigma j)p \).

Given a map \( f: A \times B \to X \) we can now define \( J(f) = (\Sigma f)d: \Sigma(A \sqcup B) \to \Sigma X \). We shall call \( J(f) \) the map obtained from \( f \) by the Hopf construction. The element \( [J(f)] \) is the unique element satisfying \( [\Sigma f] = (\Sigma q)^{\#}[J(f)] + [\Sigma(fj)p_A] + [\Sigma(fj)p_B] \) where \( p_A, p_B: A \times B \to A \sqcup B \) are induced by the projections onto the first and second coordinates respectively. We can now consider the element \( J(c'): \Sigma(X \setminus X) \to \Omega(\Sigma X \setminus \Sigma X) \). We have \( J(c')(\Sigma q \triangle), \Sigma(c' \triangle): \Sigma X \to \Sigma(\Sigma X \setminus \Sigma X) \). Let \( e: \Sigma(\Sigma X \setminus \Sigma X) \to \Sigma X \setminus \Sigma X \) be such that \( \tau(e) = 1_{\Sigma(\Sigma X \setminus \Sigma X)} \). Let \( c': \Sigma X \to \Sigma X \setminus \Sigma X \) be the cocommutator map. It is now easily checked that \( e\Sigma(\tau(c')) = c' \). Since \( c' \triangle = \tau(c') \). Theorem 4 follows immediately from

**Theorem 6.** \( \Sigma(c' \triangle) = J(c')(\Sigma q \triangle): \Sigma X \to \Sigma(\Sigma X \setminus \Sigma X) \).

**Proof.** The proof is completely dual to that of Theorem 5, and we shall omit it.

**Remark 1.** In [3], it is shown that we can find maps \( a', b' \) such that \( b'a' = 1, a'e'q = c' \). It is easily seen that \( J(c') = \Sigma(a'e') \).

**Remark 2.** Theorems 3 and 4 give other conditions for nil \( X \leq 1 \), conil \( X \leq 1 \) respectively, namely whenever some combination of factors in the factorizations of \( c, c' \) is null-homotopic.

**References**


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