A NOTE ON FINITE GROUPS IN WHICH
NORMALITY IS TRANSITIVE

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1. Introduction. We will say that a group $G$ satisfies the condition $C_p$ (where $p$ is a prime) if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in the normalizer of $P$. Here we wish to consider the relation between the condition $C_p$ and the class $3$ of all groups in which normality is a transitive relation. More precisely $G \in 3$ if and only if $H \triangleleft K \triangleleft G$ always implies that $H \triangleleft G$. Our object here is to prove

**Theorem 1.** A finite group which satisfies $C_p$ for all $p$ is a soluble $3$-group.

Let $\bar{3}$ denote the class of all groups $G$ for which $H \triangleleft K \triangleleft L \leq G$ always implies that $H \triangleleft L$: in short $\bar{3}$ is the largest subclass of $3$ that is closed with respect to forming subgroups. Now every finite soluble $3$-group is a $\bar{3}$-group [2, Satz 4] and it is obvious that a finite $\bar{3}$-group satisfies $C_p$ for all $p$, since every subgroup of a finite $p$-group is subnormal. Consequently we have

**Theorem 1*.** If $G$ is a finite group, the following are equivalent statements.

(i) $G$ is soluble $3$-group.
(ii) $G$ is a $\bar{3}$-group.
(iii) $G$ satisfies $C_p$ for all $p$.

Every soluble $3$-group is metabelian [4, Theorem 2.3.1], so Theorem 1* yields the following information about infinite $\bar{3}$-groups.

**Corollary.** A locally finite $\bar{3}$-group is soluble.

The proof of Theorem 1 uses the Schur-Zassenhaus splitting theorem, Burnside's theorem on the existence of a normal complement of a Sylow subgroup that lies in the centre of its normalizer and Grün's First Theorem [3]. In addition we need some simple facts about $3$-groups, the first of which has already been mentioned.

(A) Soluble $3$-groups are metabelian.
(B) Let $N \triangleleft G$ where every subnormal subgroup of $N$ is normal in $G$, $G/N$ belongs to $3$ and the order of $N$ is prime to its index. Then $G$ belongs to $3$ [4, Lemma 5.2.2].

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1 However, not every finite $\bar{3}$-group or infinite soluble $\bar{3}$-group is in $\bar{3}$, [4].
(C) If $A$ is a finite abelian group and $\alpha$ is a \textit{power automorphism} of $A$ (i.e. an automorphism which leaves every subgroup of $A$ invariant), then there exists a positive integer $m$ such that $a^\alpha = a^m$ for all $a \in A$, \cite[p. 88]{2}.

2. $c_p$ and $p$-nilpotence. We will prove Theorem 1 via two preliminary results connecting $c_p$ with the notion of $p$-nilpotence. (Recall that a finite group $G$ is $p$-\textit{nilpotent} if it has a normal subgroup of index a power of $p$ and of order prime to $p$.)

**Theorem 2.** If the finite group $G$ satisfies $c_p$ where $p$ is the smallest prime dividing the order of $G$, then $G$ is $p$-nilpotent.

**Proof.** Let $P$ be a Sylow $p$-subgroup of $G$ and let $N = N_\sigma(P)$, its normalizer in $G$. Every subgroup of $P$ is normal in $N$, so $P$ is either abelian or Hamiltonian. Suppose that $p$ is odd, so that $P$ is abelian. Elements of $N$ induce power automorphisms in $P$ of order prime to $p$ and therefore dividing $p - 1$. Since $p$ is the smallest prime dividing the order of $G$, $P$ lies in the centre of $N$ and by Burnside's theorem $P$ has a normal complement, i.e. $G$ is $p$-nilpotent. Now let $p = 2$. By the Schur-Zassenhaus theorem $N$ splits over $P$ and so we may write $N = PH$, with $P \cap H = 1$. A power automorphism of $P$ has order a power of 2 (whether or not $P$ is abelian) and $H$ has odd order. Hence $H$ centralizes $P$ and $N = P \times H$.

If $P$ is abelian, it is central in $N$ and we can again use Burnside's theorem. Suppose therefore that $P$ is Hamiltonian.

Let $\sigma$ denote the transfer of $G$ into $P/P'$ and let $K = \text{Ker } \sigma$. Then

$$G/K \cong P/P'$$

where

$$P^* = (P \cap N') \prod_{\varphi \in G} (P \cap (P')^\varphi)$$

by Grün's First Theorem. $P \cap (P')^\varphi$ and $(P \cap (P')^\varphi)^{-1}$ are both normal in $P$ and hence by a standard "Sylow" argument \cite[Lemma 14.3.1]{3} these subgroups are conjugate in $N$. By $c_p$

$$P \cap (P')^\varphi = (P \cap (P')^\varphi)^{-1}$$

and therefore $P \cap (P')^g \leq P'$ for all $g \in G$. On the other hand since $N = P \times H$, $P \cap N' = P'$. Thus we conclude that $P^* = P'$, which by the structure of Hamiltonian groups \cite[Theorem 12.5.4]{3} has order 2. Since $G/K \cong P/P'$, $P'$ is a Sylow 2-subgroup of $K$ and clearly it must lie in the centre of $N_K(P')$. Therefore $P'$ has a normal complement in $K$. 

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and this is obviously also a normal complement for $P$ in $G$. This completes the proof.

The solubility of groups of odd order yields the following

**Corollary.** A finite group satisfying $e_3$ is soluble.

The alternating group of degree 5 satisfies $e_3$ and $e_6$ but not $e_4$, so the hypothesis that $p$ is "smallest" cannot be omitted from the statement of Theorem 2. Let us say that a group is $p$-perfect if it has no nontrivial abelian $p$-factor groups. A finite group that is both $p$-nilpotent and $p$-perfect has order prime to $p$, so that these properties represent extremes of behaviour for finite groups. It is therefore interesting that the condition $e_p$ forces a finite group to one of those extremes. This is

**Theorem 3.** If the finite group $G$ satisfies $e_p$, $G$ is either $p$-nilpotent or $p$-perfect.

**Proof.** Let $P$ be a Sylow $p$-subgroup of $G$ and let $N = N_G(P)$. If $P$ lies in the centre of $N$, $G$ is $p$-nilpotent, so we can suppose that $[P, x] \neq 1$ for some $x \in N$. In addition we may assume that $p > 2$, otherwise $G$ would again be $p$-nilpotent, by Theorem 2. Hence $P$ is abelian and by (C) there is an $m > 0$ such that $a^x = a^m$ for any $a \in P$. If $m \equiv 1 \mod p$, $m^{p^i} \equiv 1 \mod p^{i+1}$ for $i \geq 0$, and the automorphism induced in $P$ by $x$ would have order a power of $p$; since $|N:P|$ is prime to $p$, this automorphism would have to be trivial. Hence $m \not\equiv 1 \mod p$ and so

$$\langle [a, x] \rangle = \langle a^{m-1} \rangle = \langle a \rangle$$

for all $a \in P$. Consequently $P \leq G'$, which shows that $G$ is $p$-perfect.

**Proof of Theorem 1.** Suppose that $G$ is a finite group of least order such that $G$ satisfies $e_p$ for all $p$ and yet $G$ is not a soluble 3-group. Let $p$ be the smallest prime dividing the order of $G$, so that $G$ is $p$-nilpotent by Theorem 2. We can write $G = PH$ where $H \triangleleft G$, $H \cap P = 1$ and $P$ is a Sylow $p$-subgroup of $G$. The order of $H$ is prime to its index in $G$, so $H$ satisfies all the conditions $e$ and is a soluble 3-group by minimality of $G$; thus $G$ is certainly soluble and to obtain a contradiction we have only to show that $G$ belongs to 3. Suppose that $H$ is nilpotent and hence abelian (being of odd order). Let $q$ be a prime dividing the order of $H$; then $Q$, the $q$-primary component of $H$, is the unique Sylow $q$-subgroup of $G$ and hence each subgroup of $Q$ is normal in $G$, by $e_q$. $G$ splits over $Q$ with, say, $G = KQ$ and $K \cap Q$

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2 See [6, p. 59] for a special case of this result.
$=1$; $K$ satisfies all the conditions $C$ and is therefore a 3-group, so that $G/Q$ is a 3-group. That $G$ belongs to 3 now follows by (B). Next we suppose that $H$ is non-nilpotent, so there is a prime $q$ dividing the order of $H$ such that $H$ is not $q$-nilpotent. By (A) $H'$ is abelian and Theorem 3 shows that $H$ is $q$-perfect and so $H/H'$ has order prime to $q$. Hence $Q$, the $q$-primary component of $H'$, is the unique Sylow $q$-subgroup of $G$ and has each of its subgroups normal in $G$. $G$ splits over $Q$ and $G/Q$ belongs to 3 by minimality. Finally $G$ belongs to 3 by (B) as before.

3. Pronormal subgroups. A subgroup $H$ of a group $G$ is said to be pronormal in $G$ if for any $g \in G$, $H$ and $H^g$ are already conjugate in their join $\langle H, H^g \rangle$. There is a link between pronormality and the condition $C_p$ which was pointed out by Dr. J. S. Rose (to whom the author is indebted for several useful comments).

Lemma (J. S. Rose). A finite group $G$ satisfies $C_p$ if and only if every $p$-subgroup is pronormal in $G$.

Proof. Assume that $G$ satisfies $C_p$ and let $P_0$ be any $p$-subgroup of $G$. Let $g \in G$; we show that $P_0$ and $P_0^g$ are conjugate in $J = \langle P_0, P_0^g \rangle$. Let $P_1$ be a Sylow $p$-subgroup of $J$ containing $P_0$. Then for some $x \in J$, $P_0^g \leq P_1^x$; hence $P_0$ and $P_0^{g^{-1}x}$ are both contained in $P_1$. Let $P$ be a Sylow $p$-subgroup of $G$ containing $P_1$. By $C_p$, $P_0$ and $P_0^{g^{-1}x}$ are both normal in $P$ and hence are conjugate in $N_G(P)$; by $C_p$ again, $P_0 = P_0^{g^{-1}x}$ and so $P_0^g = P_0$. To prove the converse note that pronormality and subnormality together imply normality.

Corollary. For finite groups the condition $C_p$ is inherited by subgroups and homomorphic images.

Proof. The subgroup part is clear. Let $H/N$ be a $p$-subgroup of $G/N$ where $G$ satisfies $C_p$ and let $P$ be a Sylow $p$-subgroup of $H$. By comparison of orders $H = PN$. $P$ is a Sylow $p$-subgroup of $H$. By $C_p$ implies that $H/N$ is pronormal in $G$ and clearly this implies that $H/N$ is pronormal in $G/N$. It will be noted that in the proof of Theorem 1 the fact that $C_p$ passes to subgroups was used only in situations where this was obvious.3 Observe also that the main theorem can be formulated thus:

a finite group in which for each $p$ every cyclic $p$-subgroup is pronormal is a soluble 3-group. On the other hand it is known that in a soluble 3-group all subgroups are pronormal [5].

3 This property may be combined with the well-known theorem of Frobenius on $p$-nilpotent groups [7, Theorem iv. 5.c] to give another proof of Theorem 2.
The above corollary allows us to draw a further conclusion from Theorem 3.

**Corollary (to Theorem 3).** A finite group which satisfies $C_p$ and is $p$-soluble has $p$-length $\leq 1$.

For if $G$ is such a group every subgroup of $G$ is either $p$-nilpotent or $p$-perfect and this excludes the possibility of a subgroup of $p$-length 2. (This result also follows from the main theorem of [1].)

On the other hand such a group need not be $p$-nilpotent, as the symmetric group of degree 4 with $p = 3$ shows.

**References**


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