A SOLUTION OF HADAMARD'S PROBLEM FOR A RESTRICTED CLASS OF OPERATORS[1]

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The following note deals with the Hadamard problem, which is to determine all linear 2nd order differential operators for which Huygens' principle is valid in the sense of "Hadamard's minor premise" ([1, p. 54.], see also [2, §1]). Examples of Huygens' operators are the ordinary wave operators

$$\Box_n = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial t^2}$$

in an odd number $n-1 \geqq 3$ of space dimensions and those operators equivalent[2] to $\Box_n$. These are the so-called *trivial* operators, and the famous "Hadamard conjecture"[3] claims that all Huygens' operators are trivial. Examples to the contrary show that this conjecture cannot be completely true (see [2]–[7]); it is, however, valid for real Huygens' operators with constant principal part in 4 independent variables. This result, which was first established by Matthisson [8], plays a decisive role in our present considerations.

In this note we will show that there is a sense in which Hadamard's conjecture is essentially correct for operators of the form

$$(1) \qquad L_n = \Delta_{n-1} - \frac{\partial^2}{\partial t^2} + c(t), \qquad \Delta_{n-1} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$$

with analytic $c(t)$. The connection between such Huygens' operators and $\Box_n$ is through the medium of certain *nontrivial* transformations of one Huygens' operator into another. These are the l-transforms first defined in [2] which, for convenience, we briefly recall here.

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[2] Two operators are equivalent if each can be transformed to the other through (i) proper coordinate transformations, (ii) multiplication of the dependent variable by a nonzero scalar function and (iii) multiplication of the operator by a nonzero function. These three transformation groups, which preserve the Huygens' character of an operator, are called *trivial transformations*. No operator of the form (1) can be trivial unless $c(t) \equiv 0$ (see [5]).

[3] This conjecture was in fact never asserted by Hadamard. (See Courant and Hilbert [9, p. 438].)

Let L_n be an operator of the form (1) with $c(t)$ analytic and $\not\equiv 0$ on an interval I, and let $\mu(t)$ be a nonzero solution of

$$\ddot{\mu} + c(t)\mu = 0 \qquad (\cdot = d/dt) \tag{2}$$

on an interval $I' \subset I$. If we define the operators

$$l_\mu = \partial/\partial t - \dot{\mu}/\mu, \qquad l_\mu{}^* = -\partial/\partial t - \dot{\mu}/\mu,$$

we may write for $t \in I'$, $L_n = \Delta_{n-1} + l_\mu^* l_\mu$. Therefore $l_\mu L_n = \tilde{L}_n l_\mu$ where $\tilde{L}_n = \Delta_{n-1} + l_\mu l_\mu^*$ is called the l_μ-transform (on I') of L_n. In [2], Theorem 1, the following fundamental result was established.

THEOREM 1. *Let L_n be an analytic Huygens' operator of the form* (1) *and $\tilde{L}_n$ be the l_μ-transform of L_n. Then $\tilde{L}_{n+2}$ is also a Huygens' operator.*

Since for even $n \geqq 4$, $\Box_n$ is a Huygens' operator, one may invoke this result to generate an infinite number of classes of nontrivial Huygens' operators.

The operator L_n will be called *l-equivalent to* $\Box_n$ if L_n can be mapped to $\Box_n$ by a finite number of l-transforms, that is, if for some nonnegative integer q we have

$$l_{\mu_0} l_{\mu_1} \cdots l_{\mu_q} L_n = \Box_n l_{\mu_0} l_{\mu_1} \cdots l_{\mu_q} \tag{3}$$

where

$$\ddot{\mu} + c(t)\mu_q = 0, \quad (l_{\mu_k} l_{\mu_k}^*)\mu_{k-1} = 0, \qquad k = q, q-1, \cdots, 1. \tag{4}$$

Our principal result may now be stated as

THEOREM 2. *Every Huygens' operator L_n of the form* (1) *with analytic $c(t)$ is l-equivalent to $\Box_n$.*

A partial converse of Theorem 2 follows as a corollary of Theorem 1, namely, if L_n is l-equivalent to $\Box_n$ through (3) and (4), then L_n must be a Huygens' operator for even $n \geqq 2(q+3)$. In fact the relation $l_\mu L_n = \tilde{L}_n l_\mu$ implies $l_{1/\mu} \tilde{L}_n = L_n l_{1/\mu}$ where $1/\mu$ satisfies $(l_\mu l_\mu^*)1/\mu = 0$. Now set $L_n = L_n^{q+1}$ and denote by L_n^{q-k} the $l_{\mu_{q-k}}$-transform of L_n^{q-k+1}, $k = 0, 1, \cdots, q$. Then $\Box_n = L_n^0$ and L_n^{q-k+1} is the $l_{\lambda_{q-k}}$-transform of L_n^{q-k} where $\lambda_{q-k} = 1/\mu_{q-k}$. We therefore have $l_{\lambda_0} \Box_n = L^1 l_{\lambda_0}$ and more generally

$$l_{\lambda_k} \cdots l_{\lambda_0} \Box_n = L_n^{k+1} l_{\lambda_0} \cdots l_{\lambda_k}, \qquad k = 0, \cdots, q. \tag{5}$$

Setting $k = q$ in (5), it then follows from Theorem 1 that L_n must be a Huygens' operator for even $n \geqq 4 + 2(q+1) = 2(q+3)$.

If L_n is a Huygens' operator and $\tilde{L}_n$ its l_μ-transform, it is not generally true that $\tilde{L}_n$ is a Huygens' operator. However, one can establish the following result, upon which is based the proof of the theorem.

LEMMA 1. *Let L_n, $n \geq 6$, be a Huygens' operator of the form* (1) *with analytic $c(t)$. There exists $\mu(t) \neq 0$ satisfying* (2) *such that*

(A) *if $\tilde{L}_n$ is the l_μ-transform of L_n, then $\tilde{L}_{n-2}$ is a Huygens' operator.*

Moreover, $\mu(t)$ is uniquely determined by condition (A) *up to a constant factor.*

The proof of this lemma rests heavily on the Hadamard theory of elementary solution [1]. For the relevant facts of the theory we refer to [2, § 2]. In particular we make use of the fact that if our operator L_n is Huygens', the same is true for L_{n+2m}, $m=1, 2, \cdots$. Since the l_μ-transforms do not depend on n, it follows that if Lemma 1 (or Theorem 2) holds for a certain value of n, it holds also for $n+2$, $n+4, \cdots$. We may therefore assume in our proofs, and we do so, that n is such that L_{n-2} is not a Huygens' operator.

PROOF OF THEOREM 2. Assuming for the moment the validity of the lemma, let L_n be a given Huygens' operator with analytic $c(t)$. If $n=4$, then we know from [8] that $L_n = \Box_n$. For $n=6, 8, \cdots$, set $q=(n-6)/2$. According to the lemma, there is a function $\mu_q(t)$ satisfying (2) such that L_n^q, the l_{μ_q}-transform of L_n, has the property that L_{n-2}^q is a Huygens' operator. Explicitly we have

$$L_n^q = \Delta_{n-1} + l_{\mu_q} l_{\mu_q}^*.$$

If $n-2=4$, the proof is finished. Otherwise applying the lemma next to L_n^q, we obtain a function $\mu_{q-1}(t)$ satisfying $(l_{\mu_q} l_{\mu_q}^*)\mu_{q-1}=0$ such that the $l_{\mu_{q-1}}$-transform of L_n^q,

$$L_n^{q-1} = \Delta_{n-1} + l_{\mu_{q-1}} l_{\mu_{q-1}}^*$$

is still a Huygens' operator if Δ_{n-1} is replaced by Δ_{n-5}. Composing these two l-transforms gives the identity

$$l_{\mu_{q-1}} l_{\mu_q} L_n = l_{\mu_{q-1}} L_n^q l_{\mu_q} = L_n^{q-1} l_{\mu_{q-1}} l_{\mu_q}.$$

Continuing in this fashion, we obtain a sequence of operators l_{μ_k}, $k=q, q-1, \cdots, 0$, defined by

$$\ddot{\mu}_q + c(t)\mu_q = 0, \quad (l_{\mu_k} l_{\mu_k}^*)\mu_{k-1} = 0, \qquad k = q, q-1, \cdots, 1,$$

such that

$$l_{\mu_0} l_{\mu_1} \cdots l_{\mu_q} L_n = L_n^0 l_{\mu_0} l_{\mu_1} \cdots l_{\mu_q}$$

and such that L_4^0 is a Huygens' operator. Again from [8] we conclude that $L_4^0 = \Box_4$ and thus $L_n^0 = \Box_n$, proving the theorem.

PROOF OF LEMMA 1. Let L_n, $n \geqq 6$ be a fixed Huygens' operator with $c(t)$ analytic on the interval I. In this case, the elementary solution (in the sense of Hadamard) of $L_n u = 0$ relative to a point $\tau \in I$, $\xi \in \mathbb{R}^{n-1}$, is of the form

$$\Gamma^{-(p+1)} \sum_{\nu=0}^{p} \beta_\nu(n) U_\nu(t, \tau) \Gamma^\nu + R, \qquad p = \frac{n-4}{2},$$

where $\Gamma = \sum_{i=1}^{n-1} (x_i - \xi_i)^2 - (t-\tau)^2$, R is some regular function and

$$\beta_0 = 1, \qquad \beta_\nu(n) = \alpha_1 \alpha_2 \cdots \alpha_\nu,$$
$$\alpha_\nu = 1/4(p - \nu + 1), \qquad \nu \neq p + 1, \qquad \alpha_{p+1} = \tfrac{1}{4}.$$

The regular functions $U_\nu(t, \tau)$ are uniquely determined throughout I and depend only on the form of $c(t)$ and not on n. In particular, $U_p(t, \tau)$ is a solution of $L_n u = 0$ and hence satisfies (2) identically in $(t, \tau) \in I \times I$. We assume $U_p(t, \tau) \not\equiv 0$, which is equivalent to the assumption that L_{n-2} is not a Huygens' operator.

The construction of the solution $\mu(t)$ is via the coefficient $U_p(t, \tau)$. In fact, since U_p satisfies (2) the l_{U_p}-transform, say $\tilde{L}_n$, of L_n is defined. Part (A) of Lemma 1 is established by proving first of all that $U_p(t, \tau)$ has the form[4] $U_p(t, \tau) = (\text{const})\, a_p(t) a_p(\tau)$ (so that $\tilde{L}_n$ does not depend on τ). We then verify that $\tilde{L}_{n-2}$ defined in this way (or equivalently, via the solution $a_p(t)$) is indeed a Huygens' operator. The verification of the formula above for $U_p(t, \tau)$ constitutes the main part of the proof.

As the $U_\nu(t, \tau)$ are independent of n, they may be used to form the elementary solution of $L_m u = 0$ for any integer $m \geqq 2$. In particular, for $m = 2$ the elementary solution is $W \log \bar{\Gamma} + \bar{R}$ where

$$(6) \qquad \begin{aligned} W(t, \tau, r) &= \sum_{\nu=0}^{p} \beta_\nu U_\nu(t, \tau)(r^2 - T^2)^\nu, \\ \beta_\nu &= \beta_\nu(2), \qquad \bar{\Gamma} = r^2 - T^2 = (x_1 - \xi_1)^2 - (t - \tau)^2. \end{aligned}$$

Hadamard has shown [1, pp. 71, 72] that W is identical with the regular Riemann function of L_2. Thus W is a regular solution of $L_2 u = 0$.

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[4] The fact that L_n is formally selfadjoint implies the symmetry relation $U_\nu(t, \tau) = U_\nu(\tau, t)$. Since moreover U_p satisfies (2), it must have the form $U_p(t, \tau) = k_{11}\mu(t)\mu(\tau) + k_{12}(\mu(t)\lambda(\tau) + \mu(\tau)\lambda(t)) + k_{22}\lambda(t)\lambda(\tau)$ where (μ, λ) is any linearly independent pair of solutions of (2) and (k_{ij}) is a symmetric 2×2 matrix of constants. The statement $U_p(t, \tau) = k a_p(t) a_p(\tau)$ is equivalent to rank $(k_{ij}) = 1$.

We now define for real values of ρ the function

$$\hat{u}(t, \tau, \sigma) = \frac{\eta(T)}{2}\int_{-T}^{T} W(t, \tau, r)e^{-2\pi i\rho r}\, dr, \qquad \sigma = \frac{1}{2\pi i\rho} \tag{7}$$

where $\eta(T)$ is equal to 1 or zero depending on whether $T \geqq 0$ or <0. Then $\hat{u}$ is the fundamental solution for the ordinary differential operator

$$\hat{L} = d^2/dt^2 + (c(t) - 1/\sigma^2)$$

that is, $\hat{u}$ satisfies

(a) $d\hat{u}/dt$ and $d^2\hat{u}/dt^2$ are continuous with respect to t in I except at $t = \tau$, where $(d^+/dt - d^-/dt)\hat{u} = 1$,

(b) $\hat{L}\hat{u} = 0$ in I except at $t = \tau$,

(c) $\hat{u} \equiv 0$ for $t < \tau$.

The conditions (a), (b) and (c) imply the uniqueness of the fundamental solution. Replacing W in (7) through (6) gives

$$\hat{u}(t, \tau, \sigma) = \tfrac{1}{2}\eta(T) \sum_{\nu=0}^{p} \beta_\nu U_\nu(t, \tau) \int_{-T}^{T} (r^2 - T^2)^\nu e^{-2\pi i\rho r}\, dr.$$

The integral can be expressed as

$$\begin{aligned}\int_{-T}^{T} (r^2 - T^2)^\nu e^{-2\pi i\rho r}\, dr &= (-1)^\nu T^{\nu+1/2} \frac{\nu!\pi^{1/2}}{(\pi\rho)^{\nu+1/2}} J_{\nu+1/2}(2\pi\rho T) \\ &= 2\sigma \operatorname{Im}\left(e^{T/\sigma} \sum_{k=0}^{\nu} \overset{\nu}{A}_k T^{\nu-k} \sigma^{\nu+k}\right)\end{aligned}$$

where $\operatorname{Im}(z)$ = imaginary part of z, $J_{\nu+1/2}$ is the Bessel function of first kind of order $\nu + 1/2$ and

$$\overset{\nu}{A}_k = \frac{(-1)^{\nu+k}\nu!(\nu+k)!}{(\nu-k)!} 2^{\nu-k}, \qquad k = 0, 1, \cdots, \nu.$$

Thus

$$\begin{aligned}\hat{u}(t, \tau, \sigma) &= \sigma\eta(T) \sum_{\nu=0}^{p} \beta_\nu U_\nu(t, \tau) \operatorname{Im}\left(e^{T/\sigma} \sum_{k=0}^{\nu} \overset{\nu}{A}_k T^{\nu-k} \sigma^{\nu+k}\right) \\ &= \sigma\eta(T) \operatorname{Im}\left(e^{T/\sigma} \sum_{\nu=0}^{2p} \hat{U}_\nu(t, \tau)\sigma^\nu\right)\end{aligned} \tag{8}$$

with

$$\hat{U}_{2\nu}(t, \tau) = \sum_{k=0}^{\nu} \beta_{\nu+k} \overset{\nu+k}{A}_{\nu-k} U_{\nu+k}(t, \tau) T^{2k}, \qquad \nu = 0, \cdots, p,$$

$$\hat{U}_{2\nu+1}(t, \tau) = T \sum_{k=0}^{\nu} \beta_{\nu+k+1} \overset{\nu+k+1}{A}_{\nu-k} U_{\nu+k+1}(t, \tau) T^{2k}, \qquad \nu = 0, \cdots, p-1.$$

In particular, since $U_\nu \equiv 0$ for $\nu > p$,

$$\hat{U}_{2p}(t, \tau) = \beta_p A_p^p U_p(t, \tau) \not\equiv 0.$$

We next show that $\hat{U}_{2p}(t, \tau) = a_p(t)a_p(\tau)$ where $a_p(t)$ is some solution of (2). In view of (8), $\hat{L}u = 0$ admits a solution in I of the form $u(t, \sigma) = e^{t/\sigma} \sum_{\nu=0}^{r} b_\nu(t)\sigma^\nu$ where the b_ν are real and $b_r(t) \not\equiv 0$. We shall call the integer r the *degree* of $u(t, \sigma)$. Let

$$u_m(t, \sigma) = e^{t/\sigma} \sum_{\nu=0}^{q} a_\nu(t)\sigma^\nu \equiv e^{t/\sigma} P_m(t, \sigma) \tag{9}$$

be the *minimal solution* of $\hat{L}u = 0$, that is, the solution of least degree. The coefficients $a_\nu(t)$ of P_m satisfy

$$\dot{a}_0 = 0, \quad \dot{a}_\nu = -\tfrac{1}{2}(\ddot{a}_{\nu-1} + c(t)a_{\nu-1}), \qquad \nu \geqq 1.$$

It is easy to verify that every solution in I of the form $e^{t/\sigma} \sum_{\nu=0}^{r} b_\nu(t)\sigma^\nu$ is a multiple of u_m by a polynomial in σ with constant coefficients. With the normalization $a_0 \equiv 1$, the minimal solution is clearly unique. We note that the coefficient $a_q(t)$ in u_m satisfies $\ddot{a}_q + c(t)a_q = 0$.

The fundamental solution $\hat{u}$ of $\hat{L}u = 0$ may be expressed in terms of u_m as

$$\begin{aligned} \hat{u}(t, \tau, \sigma) &= \eta(T) \frac{u_m(t, \sigma)\bar{u}_m(\tau, \sigma) - \bar{u}_m(t, \sigma)u_m(\tau, \sigma)}{\mathcal{W}(u_m, \bar{u}_m)} \\ &= 2n(T) \operatorname{Im}[e^{T/\sigma} P_m(t, \sigma)\overline{P}_m(\tau, \sigma)]/\mathcal{W}(u_m, \bar{u}_m) \end{aligned} \tag{10}$$

where $\bar{u}_m$ = complex conjugate of u_m and the Wronskian

$$\begin{aligned} \mathcal{W}(u_m, \bar{u}_m) &= \bar{u}_m(t, \sigma) \frac{du_m}{dt}(t, \sigma) - u_m(t, \sigma) \frac{d\bar{u}_m}{dt}(t, \sigma) \\ &= \frac{2}{\sigma}\left[P_m\overline{P}_m + \frac{\sigma}{2}\left(\overline{P}_m \frac{dP_m}{dt} - P_m \frac{d\overline{P}_m}{dt}\right)\right] \end{aligned} \tag{11}$$

is independent of t. The expression in brackets in (11) is both real and a polynomial in σ, and therefore

$$\mathcal{W}(u_m, \bar{u}_m) = \frac{2}{\sigma} \sum_{\nu=0}^{q} k_{2\nu}\sigma^{2\nu} \tag{12}$$

with $k_0 = 1$, $k_{2\nu}$ = const, $\nu = 1, \cdots, q$. On account of the uniqueness of $\hat{u}$ it follows from (8), (10) and (12) that

$$\sum_{\nu=0}^{2p} \hat{U}_\nu(t, \tau)\sigma^\nu = P_m(t, \sigma)\overline{P}_m(\tau, \sigma) \Big/ \sum_{\nu=0}^{q} k_{2\nu}\sigma^{2\nu}.$$

Since u_m is the minimal solution of $\hat{L}u=0$, neither P_m nor $\overline{P}_m$ can admit a polynomial divisor with constant coefficients and so we must have

$$
\begin{aligned}
&q = p, \quad k_{2\nu} = 0, \qquad \nu = 1, 2, \cdots, p, \\
(13) \qquad &\hat{U}_{2p}(t, \tau) = a_p(t)a_p(\tau) = \beta_p A_p^p U_p(t, \tau) \not\equiv 0.
\end{aligned}
$$

The formula (13) establishes the representation of the coefficient U_p which was mentioned at the beginning of the proof. Moreover, we have proved the following:

LEMMA 2. *Let $L_n=\Box_n+c(t)$ be a Huygens' operator with analytic $c(t)$. Then the equation $\hat{L}u=\ddot{u}+(c(t)-1/\sigma^2)u=0$ admits a solution of the form*

$$u(t, \sigma) = e^{t/\sigma} \sum_{\nu=0}^{p} a_\nu(t)\sigma^\nu, \qquad p = \frac{n-4}{2}\cdot$$

If, moreover, L_{n-2} is not a Huygens' operator, then $u(t, \sigma)$ is the minimal solution of $\hat{L}u=0$.

To complete the proof of (A) of Lemma 1, set $\mu(t)=a_p(t)$ and let $\tilde{L}_n$ be the l_μ-transform of L_n. We will show that $\tilde{L}_{n-2}$ is a Huygens' operator.

Let m denote the smallest number of independent variables in which $\tilde{L}_m$ is a Huygens' operator. As L_{n-2} is not a Huygens' operator but is the $l_{1/u}$-transform of $\tilde{L}_{n-2}$, from Theorem 1 follows that $m \geqq n-2$. To prove $m=n-2$ it suffices, by Lemma 2, to show that $\check{L}u=-(l_\mu l_\mu^*+1/\sigma^2)u=0$ admits a solution of degree $p-1$.

But from $l_\mu L_n=\tilde{L}_n l_\mu$ we evidently have

$$l_\mu \hat{L} = \check{L} l_\mu$$

and therefore $\check{L}(l_\mu\, u_m(t))=0$. Explicitly

$$(14) \qquad l_\mu u_m(t) = \frac{1}{\sigma} e^{t/\sigma} \sum_{\nu=0}^{p+1} \left(a_\nu + \dot{a}_{\nu-1} - a_{\nu-1}\frac{\dot{a}_p}{a_p} \right) \sigma^\nu .$$

The coefficient of σ^{p+1} in (14) is $\dot{a}_p-a_p(\dot{a}_p/a_p)\equiv 0$. We have to show that the coefficient of σ^p also vanishes identically, i.e.

$$(15) \qquad a_p + \dot{a}_{p-1} - a_{p-1}(\dot{a}_p/a_p) \equiv 0.$$

But (15) is a special case of a general recursion formula satisfied by the coefficients a_ν in u_m. In fact, by substituting the right-hand side of

(9) into the right member of (11) and then using (12) and (13) we obtain

$$(16)\quad 2a_{2\nu} = 2\sum_{k=1}^{\nu-1}(-1)^{k+1}a_k a_{2\nu-k} + \sum_{k=0}^{2\nu-1}(-1)^{k+1}a_k\dot{a}_{2\nu-k-1} + (-1)^{\nu+1}a_\nu^2$$

valid for $\nu=1, 2, \cdots$, where $a_\nu \equiv 0$ for $\nu > p$. Setting $\nu = p$ in (16) gives immediately the identity (15). Thus $\sigma l_\mu u_m$ is a solution of $\check{L}u=0$ having degree $p-1$.

To prove the uniqueness part of Lemma 1, let $\mu(t)$ and $\lambda(t)$ be two nontrivial solutions of (2) for which condition (A) is satisfied, that is, if $\tilde{L}_n$ and $\bar{L}_n$ are the l_μ- and l_λ-transforms of L_n, respectively, then both $\tilde{L}_{n-2}$ and $\bar{L}_{n-2}$ are Huygens' operators. Since L_{n-2} is the $l_{1/\mu}$-transform of $\tilde{L}_{n-2}$, the coefficient $U_p(t, \tau)$ in the elementary solution of $L_n u = 0$ is given in terms of μ by $U_p(t,\tau) = \tilde{k}\mu(t)\mu(\tau)$, $\tilde{k} =$ const., as was shown in [2, §3]. Also, L_{n-2} is the $l_{1/\lambda}$-transform of $\bar{L}_{n-2}$ so that $U_p(t,\tau) = \bar{k}\lambda(t)\lambda(\tau)$. It follows from the uniqueness of U_p that $\tilde{k}\mu(t)\mu(\tau) = \bar{k}\lambda(t)\lambda(\tau)$, and since $U_p \not\equiv 0$ neither $\tilde{k}$ nor $\bar{k}$ can vanish. Therefore $\mu(t) = (\text{const})\lambda(t)$ and the proof of Lemma 1 is complete.

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