ON THE ENTROPY OF ISOMETRIES OF COMPACT METRIC SPACES

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It is known from spectral considerations that the entropy of an isometry T of a compact metric space with respect to a finite invariant Borel measure is zero. In this note we compute the entropy of such metric automorphisms by constructing an increasing sequence of finite σ-fields $\{\mathcal{G}_n\}$ such that $\bigvee_{j=-\infty}^{+\infty} T^j(\bigvee_{n=1}^{\infty} \mathcal{G}_n)$ is the Borel field $\mathcal{B}$ and such that the relative entropy of T given $\mathcal{G}_n$, $h(T, \mathcal{G}_n)$, is zero for each n. It then follows from a theorem of Sinai [3] that $h(T)=0$.

We now proceed to the construction of the $\mathcal{G}_n$. Let M denote a compact metric space, T an isometry of M and m a T-invariant probability measure on the Borel sets $\mathcal{B}$ of M. Given any sphere S in M with center x there exists a sphere S' with center x and radius less than that of S whose boundary, $\partial S'$, has measure zero. For if the boundaries of all spheres contained in S and centered at x had positive measure then the measure of M could not be finite since all these boundaries are disjoint and there are a nondenumerable number of such sets. Thus for each $x \in M$ and each positive integer n, there exists a sphere $S(\epsilon(n, x); x)$ with center x and radius $\epsilon(n, x) < 1/n$ such that $m(\partial S(\epsilon(n, x); x)) = 0$.

For each n, let $A_n = \{S(\epsilon(n, x_j^n); x_j^n) : j = 1, 2, \cdots, k_n\}$ be a finite collection of the spheres described above which covers M, and define $\mathcal{F}_n$ to be the σ-field generated by the sets in A_n. Define $\mathcal{G}_1$ to be $\mathcal{F}_1$ and for $n \geqq 2$, $\mathcal{G}_n$ to be the σ-field generated by $\mathcal{G}_{n-1}$ and $\mathcal{F}_n$.

We now show that $\mathcal{G} = \bigvee_{n=1}^{\infty} \mathcal{G}_n = \mathcal{B}$. Let U be an open set and $p \in U$. There is an $\epsilon > 0$ such that $S(\epsilon, p) \subset U$. Select n so large that $1/n < \epsilon/2$ and consider the open cover A_n. There is a j such that $1 \leqq j \leqq k_n$ and $p \in S(\epsilon(n, x_j^n); x_j^n)$. Since $\epsilon(n, x_j^n) < 1/n < \epsilon/2$ we have that $S(\epsilon(n, x_j^n); x_j^n) \subset S(\epsilon, p)$. Thus for each $p \in U$ there is an open set $U_p \in \mathcal{G}$ which contains p and is contained in U and it follows that $U = \bigcup\{U_p : p \in U\}$. Since M is second countable there is a countable number of the U_p whose union is U. Thus $U \in \mathcal{G}$ and since U was an arbitrary open set $\mathcal{B} \subset \mathcal{G}$. Since $\mathcal{G}$ is generated by open sets $\mathcal{G} \subset \mathcal{B}$ and we have that $\mathcal{G} = \mathcal{B}$.

THEOREM. *If M is a compact metric space, T an isometry of M, and m a T-invariant probability measure on the Borel sets $\mathcal{B}$ of M then the m-entropy of T is zero.*

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PROOF. Let $\mathcal{G}_n$ and $\mathfrak{F}_n$ be the σ-fields constructed above. From a theorem of Sinai [**3**], (see also Jacobs [**2**]), we have that $\lim_n h(T, \mathcal{G}_n) = h(T)$. We will show that for each integer n, $\mathcal{G}_n \subset \bigvee_{j=1}^{\infty} T^{-j}\mathcal{G}_n$ a.e. from which it follows that $h(T, \mathcal{G}_n) = 0$ and hence $h(T) = 0$.

Since $\mathcal{G}_n = \bigvee_{j=1}^{n} \mathfrak{F}_j$ is the same σ-field as the σ-field generated by the A_j's for $j = 1, 2, \cdots, n$ it is enough to show that $S(\epsilon(q; x_p^q); x_p^q) \in \bigvee_{j=1}^{\infty} T^{-j}\mathcal{G}_n$ a.e. for $q = 1, 2, \cdots, n$ and $p = 1, 2, \cdots, k_q$.

Let q and p be given with $1 \leqq q \leqq n$ and $1 \leqq p \leqq k_q$. Denote x_p^q by y and $\epsilon(q, x_p^q)$ by ϵ and consider the sphere $S(\epsilon; y)$. Since the closure of $\bigcup_{j=-\infty}^{+\infty} T^j(y)$ is a minimal set there exists a sequence $\{t_r\}$ of positive integers such that $t_r \rightarrow +\infty$ and $\rho(T^{-t_r}(y), y) \downarrow 0$. Since T is an isometry and sends spheres into spheres of the same radius, we have

$$S(\epsilon; y) \subset \liminf_r T^{-t_r}(S(\epsilon; y)) \subset \limsup_r T^{-t_r}(S(\epsilon; y)) \subset \operatorname{cl} S(\epsilon; y).$$

Since $m(\partial S(\epsilon; y)) = 0$ it follows that $S(\epsilon; y) = \liminf_r T^{-t_r}(S(\epsilon; y))$ a.e. and hence that $S(\epsilon; y) \in \bigvee_{j=1}^{\infty} T^{-j}\mathcal{G}_n$ a.e.

COROLLARY. *If M is a compact metric space and (M, T) is an equicontinuous dynamical system, i.e. the family $\{T^n\}$ is equicontinuous then the m-entropy of T is zero for every T-invariant Borel probability measure m.*

PROOF. If ρ is a given metric on M, then $\rho'(x, y) = \sup\{\rho(T^n x, T^n y): -\infty < n < +\infty\}$ is an equivalent metric on M and the equicontinuity of $\{T^n\}$ implies that T is an isometry with respect to ρ'.

The Theorem and Corollary were suggested by Examples 1 and 1(a) of [**1**].

BIBLIOGRAPHY

1. Roy L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

2. Konrad Jacobs, *Lecture notes on ergodic theory*, Vol. II, Aarhus Universitet, Aarhus, Denmark, 1962/1963.

3. Ja. G. Sinai, *Flows with finite entropy*, Dokl. Akad. Nauk SSSR **125** (1959), 1200–1202. (Russian)

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