ON LIFTING TRANSFORMATION GROUPS

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It is known [1] that if (X, G, μ) is a topological transformation group such that $\pi_1(G)=0$, and if $\tilde{X}$ is a covering space of X, then there is a unique topological transformation group $(\tilde{X}, G, \tilde{\mu})$ which covers (X, G, μ). In [2] the fundamental group $\sigma(X, x_0, G)$ of a group of homeomorphisms G of a topological space X is defined, and it is observed in the proof of Theorem 7 of that paper that $\sigma(X, x_0, G)$ acts in a natural way as a group of homeomorphisms of the universal covering space $\tilde{X}$. In this note the relationship between these two results is investigated, and equations defining $\tilde{\mu}$ are found.

The language and notations will be minor modifications of those of [2], together with standard notations for covering spaces. It will be assumed throughout that X is path-connected, locally path-connected, and locally simply connected, that G is a locally path-connected topological group, and that (X, G, μ) is a topological transformation group.

First observe that every homeomorphism g of a path-connected space X induces an automorphism of the group N of normal subgroups of $\pi_1(X, x_0)$. If k is a path from gx_0 to x_0, then the map $[f]\rightarrow[k\rho+gf+k]$ is an automorphism of $\pi_1(X, x_0)$. The image of a normal subgroup π_0 is a normal subgroup which is independent of k, and which will be denoted by $g_*\pi_0$. It is easily checked that $g_*: N\rightarrow N$ is an automorphism.

DEFINITION 1. A normal subgroup π_0 of $\pi_1(X, x_0)$ is said to be invariant under G if for every $g\in G$, $g_*\pi_0=\pi_0$.

Let $[f; g]_{\pi_0}$ denote the equivalence class of the path f of order g under the equivalence relation

$$(f; g)R(f'; g') \quad \text{iff } g = g' \quad \text{and} \quad [f + f'\rho] \in \pi_0.$$

LEMMA 1. *If π_0 is invariant under G, then the set of homotopy classes $[f; g]_{\pi_0}$ with the rule of composition*

$$[f_1; g_1]_{\pi_0} * [f_2; g_2]_{\pi_0} = [f_1 + g_1 f_2; g_1 g_2]_{\pi_0}$$

forms a group $\sigma_{\pi_0}(X, x_0, G)$.

The proof is omitted.

The group $\sigma(X, x_0, G)$ defined in [2] corresponds to the case in which π_0 is the trivial subgroup of $\pi_1(X, x_0)$.

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There are two short exact sequences for $\sigma_{\pi_0}(X, x_0, G)$, in which the maps correspond to those in the exact sequence for $\sigma(X, x_0, G)$ described in [2].

$$0 \to \pi_1(X, x_0), \pi_0 \xrightarrow{i} \sigma_{\pi_0}(X, x_0, G) \xrightarrow{j} G \to 0,$$

$$0 \to \pi_0 \to \sigma(X, x_0, G) \to \sigma_{\pi_0}(X, x_0, G) \to 0.$$

A basis of open neighbourhoods is defined for $\sigma_{\pi_0}(X, x_0, G)$ as follows. Given $[f; g]_{\pi_0}$ and open neighbourhoods $U(gx_0)$ in X and $V(e)$ in G, define $W[f; g]_{\pi_0}$ to be the set of classes $[f+f'; g']_{\pi_0}$ where $g'g^{-1} \in V$ and f' is a path in $U(gx_0)$ from gx_0 to $g'x_0$. Sets of the form of W constitute a basis for a topology on $\sigma_{\pi_0}(X, x_0, G)$. Observe that if the isotropy subgroup of G at x_0 is trivial, then the map $[f; g]_{\pi_0} \to [f]_{\pi_0}$ is an injection of $\sigma_{\pi_0}(X, x_0, G)$ into the covering space $\tilde{X}_{\pi_0}$.

PROPOSITION 1. *With the topology just defined, $\sigma_{\pi_0}(X, x_0, G)$ is a topological group.*

PROOF. A proof of the continuity of the product is sketched. The proof of the continuity of the inverse is similar.

Let elements $[f_1; g_1]_{\pi_0}$ and $[f_2; g_2]_{\pi_0}$ be given, and let an open neighbourhood W of their product be given in terms of an open neighbourhood $U(g_1g_2x_0)$ and an open neighbourhood $V(e)$. Then there exist neighbourhoods $U_i(g_ix_0)$, $V_i(e)$ $(i=1, 2)$, such that

(i) $V_1 \times V_1 \subset V$ and $V_2 \subset g_1^{-1}V_1g_1$;

(ii) $h \in V_1$ and $x' \in U_2$ imply $hg_1(x_0) \in U_1$, $hg_1g_2(x_0) \in U$ and $hg_1(x') \in U$;

(iii) every loop in U_1 is nullhomotopic in X;

(iv) every pair of elements in V_2g_1 can be joined by a path in V_1g_1.

Now let $g_i' \in V_2g_i$ and let f_i' be a path in U_i from g_ix_0 to $g_i'(x_0)$ $(i=1, 2)$. Let ϕ be a path in V_1g_1 from g_1 to g_1'. Define a map $\Phi: I \times I \to X$ by $\Phi(s, t) = \phi(s)f_2(t)$, and set $\Phi(s, 0) = \phi_0(s)$, $\Phi(s, 1) = \phi_1(s)$. Then $f_1' + \phi_0\rho$ is nullhomotopic and $\phi_0 + g_1'f_2 \sim g_1f_2 + \phi_1$. Hence $[f_1 + f_1' + g_1'(f_2 + f_2'); g_1'g_2']_{\pi_0} = [f_1 + g_1f_2 + \phi_1 + g_1'f_2'; g_1'g_2']_{\pi_0} \in W$.

Relative to this topology, the homomorphism

$$\lambda_*: \sigma_{\pi_0}(X, x_0, G) \to \sigma_{\pi_0}(X, x_1, G)$$

induced by a path λ from x_0 to x_1 is continuous. Moreover, if (ϕ, ψ): $(X, G, \mu) \to (X', G', \mu')$ is a homomorphism of the topological transformation groups, then the induced homomorphism

$$(\phi, \psi)_*: \sigma(X, x_0, G) \to \sigma(X', x_0', G')$$

is continuous.

If π^1 and π^2 are normal subgroups of $\pi_1(X, x_0)$ which are invariant under G, and if $\pi^1 \subset \pi^2$, then the map

$$\tilde{\mu}\colon \sigma_{\pi^1}(X, x_0, G) \times \tilde{X}_{\pi^2} \to \tilde{X}_{\pi^2}, \quad \tilde{\mu}\colon ([f; g]_{\pi^1}, [f']_{\pi^2}) \to [f + gf']_{\pi^2},$$

is well defined.

PROPOSITION 2. *With the action $\tilde{\mu}$ just defined, $(\tilde{X}_{\pi^2}, \sigma_{\pi^1}(X, x_0, G), \tilde{\mu})$ is a topological transformation group.*

PROOF. The proof of the continuity of $\tilde{\mu}$ is similar to the proof of Proposition 1.

All the natural diagrams commute. In particular, $\tilde{\mu}$ covers μ, and the action of $\tilde{\mu}_{\pi_0}(X, x_0, G)$ on any covering space derives from its action on $\tilde{X}_{\pi_0}$.

It is interesting to note that $\sigma(\tilde{X}_{\pi_0}, \sigma_{\pi_0}(X, x_0, G)) \cong \sigma(X, x_0, G)$ the map $[F; [f; g]_{\pi_0}] \to [pF; g]$ is a homomorphism, and that it is an isomorphism follows from the application of the 5-lemma to the sequences

$$0 \to \pi_1(\tilde{X}_{\pi_0}, \tilde{x}_0) \to \sigma(\tilde{X}_{\pi_0}, \sigma_{\pi_0}(X, x_0, G)) \to \sigma_{\pi_0}(X, x_0, G) \to 0,$$
$$0 \to \pi_0 \to \sigma(X, x_0, G) \to \sigma_{\pi_0}(X, x_0, G) \to 0.$$

In §9 of [2] the concept of a family of preferred paths is defined. One can confirm that X admits a family of preferred paths at x_0 if and only if the following sequence splits.

$$0 \to \pi_1(X, x_0) \xrightarrow{i} \sigma(X, x_0, G) \xrightarrow{j} G \to 0.$$

In this case, if $k\colon G \to \sigma(X, x_0, G)$ is a splitting homomorphism and $\tilde{\mu}\colon \sigma(X, x_0, G) \times \tilde{X}_{\pi_0} \to \tilde{X}_{\pi_0}$, then $\tilde{\mu}(\mathrm{id}, k)$ defines an action of G as a group of homeomorphisms of $\tilde{X}_{\pi_0}$. However, since k need not be continuous, this is no guarantee that there is a topological transformation group $(\tilde{X}_{\pi_0}, G, \mu_1)$ which covers (X, G, μ).

DEFINITION 2. The group $\sigma(X, x_0, G)$ will be said to admit a continuous split extension if there is a continuous homomorphism $k\colon G \to \sigma(X, x_0, G)$ such that $jk = \mathrm{id}$.

LEMMA 2. *If $\sigma(X, x_0, G)$ admits a continuous split extension then it is topologically the product of G and the discrete space $\pi_1(X)$.*

PROOF. Since X is locally simply connected, the topology for $\pi_1(X, x_0)$ induced by its inclusion in $\tilde{X}$ is discrete. The proof now follows from a study of the map $([f], g) \to [f + kg\rho; g]$.

The group G acts on itself by left translation, and if it is a path-connected group we can ask whether $\sigma(G, e, G)$ admits a continuous split extension.

PROPOSITION 3. *The group $\sigma(G, e, G)$ admits a continuous split extension if and only if $\pi_1(G, e)=0$.*

PROOF. The sufficiency of the condition is obvious. Since the isotropy subgroup of G at e is trivial, $\sigma(G, e, G)$ can be regarded as a subspace of the universal covering group $\tilde{G}$. Hence a map k: $G \to \sigma(G, e, G)$ can be thought of as a covering map of the identity map of G; whence $\pi_1(G)=p_*\pi_1(\tilde{G}, \tilde{e})=0$.

Now we can prove the proposition from which the Lima result will follow. Define $\mu_{x_0}: G \to x$, $\mu_{x_0}(g)=\mu(g, x_0)$.

PROPOSITION 4. *If $\sigma(X, x_0, G)$ admits a continuous split extension, and π_0 is invariant under G, then there exists a topological transformation group $(\tilde{X}_{\pi_0}, G, \mu_1)$ which covers (X, G, μ). If $\mu_{x_0*}\pi_1(G, e)\subset\pi_0$, then the covering map μ_1 is unique.*

PROOF. The existence of the continuous map μ_1 follows from the discussion above. Its uniqueness follows from standard covering theorems.

COROLLARY. *If $\pi_1(G, e)=0$ and π_0 is invariant under G, then there exists a unique topological transformation group $(\tilde{X}_{\pi_0}, G, \mu_1)$ which covers (X, G, μ).*

PROOF. Given g, let l_g be a path in G from g to e, and let $\mu_{x_0}l_g=k_g$. Then $g\to[k_g\rho; g]$ is a continuous splitting map.

The equation defining μ_1 is

$$\mu_1: (g, [f]_{\pi_0}) \to [k_g\rho + gf]_{\pi_0}.$$

That this map is well defined even if π_0 is not invariant under G follows from an argument similar to that used to prove Theorem 7 of [2]. It is continuous, the proof being similar to that of Proposition 1. Thus, subject to the standing conditions on X and G, we have the following result.

PROPOSITION 5. *If $\pi_1(G, e)=0$ and $\tilde{X}$ is any covering space of X, then there is a unique topological transformation group $(\tilde{X}, G, \mu_1)$ which covers (X, G, μ), and the action of μ_1 is defined by the equation above.*

REFERENCES

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