THE PRIME RADICAL IN A JORDAN RING

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There are several definitions of radicals for general nonassociative rings given in literature, e.g. [1], [2], and [5]. The \( u \)-prime radical of Brown-McCoy which is given in [2], is similar to the prime radical in an associative ring. However, it depends on the particular chosen element \( u \). The purpose of this paper is to give a definition for the Brown-McCoy type prime radical for Jordan rings so that the radical will be independent from the element chosen.

Let \( J \) be a Jordan ring, \( x \) be an element in \( J \); the operator \( U_x \) is a mapping on \( J \) such that \( yU_x = 2x \cdot (x \cdot y) - x^2 \cdot y \) for all \( y \) in \( J \), or, equivalently, \( U_x = 2R_x - R_x^2 \). If \( A, B \) are subsets of \( J \), \( A \cup B \) is the set of all finite sums of elements of the form \( aU_b \), where \( a \) is in \( A \) and \( b \) is in \( B \).

**Lemma 1.** Let \( P \) be a two sided ideal in \( J \). Then the following three statements are equivalent.

(a) If \( A, B \) are ideals in \( J \) and \( A \cup B \subseteq P \), then either \( A \subseteq P \) or \( B \subseteq P \).

(b) If \( A, B \) are ideals in \( J \) with \( A \cap \text{c}(P) \neq 0 \) and \( B \cap \text{c}(P) \neq 0 \), then \( A \cup B \cap \text{c}(P) \neq 0 \), where \( \text{c}(P) \) is the complement of \( P \).

(c) If \( a, b \) are in \( \text{c}(P) \), then \( [a] \cup [b] \cap \text{c}(P) \neq \emptyset \), where \([x]\) denotes the principal ideal in \( J \) generated by \( x \).

**Proof.** Obviously (a) and (b) are equivalent.

If (b) holds and \( a, b \in \text{c}(P) \), then \([a] \cap \text{c}(P) \neq \emptyset \) and \([b] \cap \text{c}(P) \neq \emptyset \). Thus, \([a] \cup [b] \cap \text{c}(P) \neq \emptyset \), i.e. (c) holds.

If (c) holds and \( A, B \) are ideals in \( J \) such that \( A \cap \text{c}(P) \neq \emptyset \) and \( B \cap \text{c}(P) \neq \emptyset \), then there exists \( a \in A \cap \text{c}(P) \) and \( b \in B \cap \text{c}(P) \). Thus \([a] \cup [b] \cap \text{c}(P) \neq \emptyset \). But \([a] \subseteq A \) and \([b] \subseteq B \), so \( A \cup B \cap \text{c}(P) \supseteq [a] \cup [b] \cap \text{c}(P) \neq \emptyset \), i.e. (b) holds.

**Definition 1.** An ideal \( P \) in \( J \) is called a prime ideal if it satisfies any one of the statements in the Lemma 1. A nonempty subset \( M \) of \( J \) is called a \( Q \)-system if whenever \( A, B \) are ideals in \( J \) such that \( A \cap M \neq \emptyset \) and \( B \cap M \neq \emptyset \) then \( A \cup B \cap M \neq \emptyset \).

If \( P \) is an ideal in \( J \), then \( \text{c}(P) = M \) is a \( Q \)-system if, and only if \( P \) is a prime ideal.

**Definition 2.** Let \( A \) be an ideal in \( J \), then \( A^Q = \{ r \in J \mid \text{any } Q \text{-system in } J \text{ containing } r \text{ meets } A \} \) is called the \( Q \)-radical of \( A \).

**Theorem 1.** If \( A \) is an ideal in \( J \), then \( A^Q \) is the intersection of all the prime ideals \( P^* \) in \( J \) which contain \( A \).

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Proof. If \( b \in A^q \) and \( P^* \) is any prime ideal which contains \( A \), then \( b \in P^* \); otherwise, there exists a \( Q \)-system \( c(P^*) \) containing \( b \) which does not meet \( A \), thus \( b \notin A^q \). Thus \( A^q \subseteq A^q P^* \).

Conversely, if \( b \notin A^q \), there exists a \( Q \)-system \( M \) such that \( b \in M \) and \( M \cap A = \emptyset \). Applying Zorn's lemma to the family of all ideals in \( J \) which contains \( A \) but does not meet \( M \), one finds a maximal element \( P \) (partial ordering being taken as the usual set inclusion). Since \( b \) is in \( M \), \( b \) is not in \( P \). Thus it remains to show that \( P \) is a prime ideal.

If \( B, C \) are ideals in \( J \) such that \( B \subseteq P \) and \( C \subseteq P \) then both \( B + P \), and \( C + P \) meet \( M \). Thus \((P+B)U(P+C) \) meets \( M \). But \((P+B)U(P+C) \) \( \subseteq BU_C + P \), thus \( BU_C \subseteq P \). Hence \( P \) is prime.

Definition 3. An ideal \( P \) in \( J \) is a semiprime ideal if for any ideal \( A \) in \( J \), \( AU_A \subseteq P \) implies \( A \subseteq P \). A nonempty subset \( M \) of \( J \) is called a \( SQ \)-system if for any ideal \( A \) in \( J \) such that \( A \cap M \neq \emptyset \), then \( AU_A \cap M \neq \emptyset \).

The proof of Lemma 1 can be easily applied here to show an ideal \( P \) in \( J \) is semiprime if, and only if, one of the following statements holds.

(a) If \( A \) is an ideal such that \( A \cap c(P) \neq \emptyset \), then \( AU_A \cap c(P) \neq \emptyset \).
(b) If \( a \in c(P) \) then \([a]\cup[a]_P \cap c(P) \neq \emptyset \).

If \( P \) is an ideal in \( J \), then \( c(P) \) is a \( SQ \)-system if, and only if, \( P \) is semiprime.

Definition 4. Let \( A \) be an ideal in \( J \), the set \( A^q = \{ r \in J \mid \) any \( SQ \)-system containing \( r \) meets \( A \} \) is called the \( SQ \)-radical of \( A \).

Theorem 2. Let \( A \) be an ideal in \( J \), then the following statements hold

(a) \( A^q = \bigcap P_* \), where \( P_* \) are taken from all semiprime ideals in \( J \) which contain \( A \).
(b) \( A^q \) is a semiprime ideal.
(c) \( A \) is semiprime if, and only if, \( A = A^q \).

Proof. (a) If \( x \in A^q \) and \( P_* \) is a semiprime ideal in \( J \) containing \( A \), then \( x \in P_* \); otherwise, \( c(P_*) \) is a \( SQ \)-system, contains \( x \) but does not meet \( A \), so \( x \notin A^q \). Thus \( A^q \subseteq \bigcap P_* \). Conversely, if \( x \in A^q \), then there exists a \( SQ \)-system \( M \) such that \( x \in M \) and \( M \cap A \neq \emptyset \). Applying Zorn's lemma to the family of ideals in \( J \) containing \( x \) but disjoint from \( M \), one finds a maximal ideal \( P_* \). It remains to show that \( P_* \) is semiprime.

If \( B \) is an ideal in \( J \) such that \( B \subseteq P_* \), then \( P_* + B \) meets \( M \). But \( M \) is a \( SQ \)-system, thus \((P_* + B)U(P_* + B) \) meets \( M \). On the other hand, \((P_* + B)U(P_* + B) \) \( \subseteq BU + P_* \), so \( BU \subseteq P_* \).

(b) It follows from (a) that \( A^q \) is an ideal in \( J \). If \( B \) is an ideal in \( J \) such that \( BU \subseteq A^q = \bigcap P_* \), then \( B \subseteq P_* \) for all semiprime ideals \( P_* \) containing \( A \). Hence \( B \subseteq \bigcap P_* = A^q \). Thus \( A^q \) is a semiprime.
(c) Since $A_Q$ is a semiprime ideal, it is the smallest semiprime ideal in $J$ containing $A$. Thus $A$ is semiprime if, and only if, $A = A_Q$.

**Lemma 2.** Let $a$ be an element in $J$ and $S$ is a $SQ$-system in $J$ containing $a$. Then there exists a $Q$-system $M$ such that $a$ is in $M$ and $M \subseteq S$.

**Proof.** We first construct a sequence $M = \{a_1, a_2, \ldots, a_n, \ldots\}$ of elements of $J$ where $a_1 = a$, $a_2 \in [a_1] \cup [a_1] \cap S$, $\ldots$, $a_{k+1} \in [a_k] \cup [a_k] \cap S$, $\ldots$. Clearly, $a \in M$ and $M \subseteq S$. It remains to show that $M$ is a $Q$-system, i.e. $[a_i] \cup [a_j] \cap S \neq \emptyset$, for all $i, j$.

Note that $a_{i+1} \in [a_i]$, so $[a_{i+1}] \subseteq [a_i]$ and hence $[a_j] \subseteq [a_i]$ if $j \geq i$. If we let $K$ be the larger of $i$ and $j$ then $a_{k+1} \in [a_k] \cup [a_k] \cap S \subseteq [a_i] \cup [a_j] \cap S$.

**Theorem 3.** For any ideal $A$ in $J$, $A^Q = A_Q$. $A^Q$ is called the prime radical of the ideal $A$.

**Proof.** Since every prime ideal is a semiprime ideal, it is clear that $A^Q = \bigcap P \supseteq \bigcap P_\ast = A_Q$.

Conversely, if $x \in A^Q$, and $S$ is a $SQ$-system containing $x$, then by Lemma 2, there exists a $Q$-system $M$ such that $x \in M$ and $M \subseteq S$ since $M$ meets $A$, $S$ meets $A$ also.

**Definition 5.** The prime radical, $R(J)$, of a Jordan ring $J$ is the prime radical of the zero ideal in $J$. A Jordan ring is $Q$-semisimple if and only if $R(J) = (0)$.

**Theorem 4.** Let $J$ be a Jordan ring and $R(J)$ be the prime radical of $J$, then $R(J/R(J)) = (0)$, i.e. $J/R(J)$ is a $Q$-semisimple ring.

**Proof.** Let $\theta: a \to \bar{a}$ be the natural homomorphism from $J$ onto $J/R(J) = \overline{J}$. It is easy to check that the image of any prime ideal in $J$ is a prime ideal in $\overline{J}$. Let $\bar{a} \in R(\overline{J})$ and $P$ be any prime ideal in $\overline{J}$. Then $a \in \overline{P} = P/R(J)$. Hence, $a \in \theta^{-1}(\overline{P}) = P$, so $a \in \bigcap P = R(J)$ and $\bar{a} = 0$.

**Definition 6.** A ring $J$ is a prime ring if, and only if, $(0)$ is a prime ideal in $J$.

Thus, a prime ring must be $Q$-semisimple, and an ideal $P$ in $J$ is prime if, and only if, $J/P$ is a prime ring.

As in the case of associative rings, one can easily prove the following two assertions.

(a) A ring $R$ is a subdirect sum of $S_i$, $i \in I$ if, and only if, for each $i \in I$, there exists a homomorphism $\phi_i$ from $R$ onto $S_i$ and that $0 \neq r \in R$ implies $\phi_i(r) \neq 0$ for at least one $i \in I$.

(b) A ring is a subdirect sum of rings $S_i$, $i \in I$, if, and only if, for each $i \in I$ there exists a two sided ideal $K_i$ in $R$ such that $R/K_i \cong S_i$ and $\bigcap K_i = (0)$.

We obtain the following two theorems. The proof is similar to that in the associative case.
Theorem 5. A necessary and sufficient condition that a Jordan ring be isomorphic to a subdirect sum of prime rings is that \( J \) is Q-semisimple.

In the presence of the descending chain condition on ideals in \( J \), one may choose a finite subset of prime ideals \( \{ P_i | i = 1, \cdots, n \} \) in \( J \) such that \( \bigcap P_i = 0 \) and \( \bigcap_{i \neq j} P_i \neq 0 \) for any \( j = 1, 2, \cdots, n \).

Theorem 6. If \( J \) is a Jordan ring with descending chain condition on prime ideals then \( J \) is Q-semisimple if, and only if, \( Q \) is a full direct sum of finite numbers of prime ideals in \( J \).

Theorem 7. Let \( A \) be an ideal in Jordan ring \( J \) and \( r \in A_Q \), then there exists a positive integer \( k \) such that \( r^k \in A \).

Proof. It is sufficient to show that if \( r \in A_Q \), then the set \( M = \{ r, r^3, r^4, \cdots, r^k, \cdots \} \) is a SQ-system.

Suppose \( C \) is an ideal in \( J \) and \( r^k \in C \cap M \), then \( r^{k+1} \in CU_C \cap M \). Thus \( M \) is a SQ-system.

Corollary. The prime radical of a Jordan ring \( J \) is a nilideal in \( J \).

Proof. If \( r \in R(J) \), then \( r^k \in (0) \).

In a general nonassociative ring \( R \), the nil radical \( N(R) \) is the maximal nilideal in \( R[1] \). As a consequence of the corollary, the prime radical of a Jordan ring is contained in the nil radical \( N(J) \).

If \( J \) is a finite dimensional Jordan algebra, every nilideal is a nilpotent ideal. Thus, \( R(J) \) is contained in the classical radical \( S(J) \), which is the maximal nilpotent ideal in \( J \).

On the other hand, in the next theorem, any nilpotent ideal in \( J \) is contained in \( R(J) \). Thus, in this case, two definitions coincide. However, we are not sure whether in general this is also the case.

Lemma 3. Let \( A \) be an ideal in \( J \). Then \( A^3 \) is an ideal of \( J \) and \( A^3 = AU_A \).

Proof. The first assertion is a direct consequence of the linearized form of the Jordan identity: \([(a \cdot b) \cdot c] \cdot x = (a \cdot b) \cdot (c \cdot x) + (a \cdot c) \cdot (b \cdot x) + (b \cdot c) \cdot (a \cdot x) - [(a \cdot x) \cdot c] \cdot b - [(b \cdot x) \cdot c] \cdot a \). The second assertion is obtained from \( 4(x \cdot y) \cdot z = 2xU_{(y,z)} + 2yU_{(x,z)} = yU_{(x,z)} - yU_z - yU_z + xU_{(y,z)} - xU_y - xU_z \in AU_A \).

Theorem 8. A Jordan ring \( J \) is Q-semisimple if and only if it contains no nonzero nilpotent ideal.

Proof. By definition \( S \) and part (c) of Theorem 2, \( J \) is Q-semisimple if and only if \( (0) = (0)_Q \). Thus \( J \) being Q-semisimple is equivalent to the ideal \( (0) \) being semiprime. If \( J \) contains a nonzero nilpotent
ideal $M$ of nilindex $t$, then there exists a positive integer $t$ such that $M^t = 0$ and $M^{t-1} \neq 0$. Thus $(0)$ is not semiprime.

Conversely, if $J$ contains no nonzero nilpotent ideal and if $(0)$ is not semiprime, then there exists a nonzero ideal $A$ such that $AU_A \subseteq 0$. Thus $A^3 = 0$ which is impossible.

**Corollary.** The $Q$-radical $R(J)$ of a Jordan ring $J$ contains all the nilpotent ideals in $J$.

**Proof.** If $M$ is a nilpotent ideal in $J$, $\overline{M}$ is the image of $M$ under the natural homomorphism from $J$ onto $J/R(J)$. Since $\overline{M}$ is a nilpotent ideal in $\overline{J}$, $(0)$ is not a semiprime ideal in $\overline{J}$. If $\overline{A}$ is a nonzero ideal in $\overline{J}$ such that $\overline{A}^3 = \overline{A}U_{\overline{A}} = (0)$, then $AU_A \subseteq R(J)$. But $R(J)$ is semiprime, so $A \subseteq R(J)$ and $\overline{A} = (0)$ which is a contradiction.

The following theorem is due to the referee.

**Theorem 9.** If a Jordan ring $J$ contains a maximal nilpotent ideal $S(J)$ then $R(J) = S(J)$.

**Proof.** Clearly $R(J) \supseteq S(J)$ by the corollary of Theorem 8. In the ring $\overline{J} = J/S(J)$ there are no nonzero nilpotent ideals by the maximality of $S(J)$. So $\overline{J}$ is $Q$-semisimple by Theorem 8.

If $r \in R(J) = \bigcap P^*$ then $r \in S(J)$. If $r \in S(J)$, its image in $\overline{J}$ under the natural homomorphism would be $\overline{r} \neq 0$, so $\overline{r} \in (0) = R(\overline{J}) = \bigcap \overline{P}^*$ and $\overline{r} \in \overline{P}^*$ for some prime ideal $\overline{P}^*$ in $\overline{J}$. Let $P^*$ be the inverse image of $\overline{P}^*$ in $J$; then $\overline{r} \in \overline{P}^*$ implies $r \in P^*$. Since $r$ is in all prime ideals in $J$, $P^*$ cannot be prime. Thus there exists ideals $A, B$ in $J$ with $A \subseteq P^*$ and $B \supseteq P^*$ but $AU_B \subseteq P^*$. Passing to the homomorphic image $\overline{A} \subseteq \overline{P}^*$, $\overline{B} \subseteq \overline{P}^*$ but $\overline{A}U_{\overline{B}} \subseteq \overline{P}^*$. This contradicts the primeness of $\overline{P}^*$.

**References**


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