LINES IN A PLANAR SPACE

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A planar space $P$ is a set $S$ together with a mapping $A$ which attaches to each triple $(p, q, r)$ of points of $S$ a real number $A(p, q, r)$ and which satisfies:

(i) If $p = q$, then $A(p, q, r) = A(r, p, q) = 0$ for every $r$.
(ii) For every $p, q, r, s$, $A(p, q, r) + A(p, q, s) + A(p, r, s) + A(q, r, s) = 0$.
(iii) For any $p, q, r, s$; if $A(p, q, r) = A(p, q, s) = 0$, then $p = q$ or $A(q, r, s) = A(p, r, s) = 0$.

For convenience we will write $pqr$ for $A(p, q, r)$ for the remainder of the paper.

The usual example of such a space is the Euclidean $n$-space with the $A$-function interpreted as the area of a triangle with vertices $p$, $q$, and $r$.

Spaces satisfying (i) and (ii) and a variety of conditions in place of (iii) have been studied by Menger [6], Blumenthal [2], Froda [3], Gähler [5] and Freese and Andalafte [4].

For $a \neq b$ points of $P$ we define $L[a, b] = \{x | abx = 0\}$. It follows readily that if $L(a, b)$ and $L(c, d)$ are distinct sets, then $L(a, b) \cap L(c, d)$ contains at most one point.

If $p \in P$ is not an element of $L(a, b)$, we define a distance for points $x, y$ of $L$ by setting $d(x, y) = pxy$.

If $x = y$, then $pxy = 0$, but $d(x, y) = pxy = 0$. If $d(x, y) = 0$, then $pxy = 0$ and, since $x$ and $y$ belong to $L(a, b)$, $xya = xyb = 0$. Now, if $x \neq y$, applying (iii) to the quadruple $\{p, x, y, a\}$ gives $pxa = pya = 0$. Application of (iii) to the quadruple $\{p, x, y, b\}$ gives $pxb = pyb = 0$. Then since $pxa = pxb = 0$, we have $pab = xab = 0$, also from (iii). However, $pab > 0$, since $p$ is not in $L(ab)$. Therefore, it must follow that $x = y$.

Since condition (iii) may be variously applied to any three distinct points of $S$ by letting $s$ and another of the symbols $p, q, r$ denote the same point, it follows that the $A$-function is symmetric. Symmetry of the distance function follows immediately. The tetrahedral inequality applied to $\{p, x, y, z\}$ gives $pxy \leq pzx + pyz + xyz$. Since $xyz = 0$, we have $d(x, y) \leq d(x, z) + d(y, z)$.

Consequently $d(x, y)$ is a metric for $L(a, b)$. The set $L(a, b)$ with metric $d$ is denoted $M_p(a, b)$.

Presented to the Society, January 25, 1967; received by the editors June 10, 1967.

1 This paper represents a portion of the author's dissertation written under the direction of Raymond W. Freese at St. Louis University.
We will utilize the following definitions.

A point $b$ is said to be between $a$ and $c$ (denoted by $B(a, b, c)$) iff $abc = 0$, $acx = abx + bcx$ for every $x$ and $a$, $b$, and $c$ are distinct.

A planar space $P$ is convex iff for each pair of different points $p$ and $q$ there exists a between point.

A sequence of points $\{x_n\}$ in a planar space $P$ has limit $x$ iff $\lim pxx_n = 0$ for every $p$ in $P$.

A sequence $\{x_n\}$ in a planar space $P$ is convergent with respect to $(a, b, c)$ iff $abc > 0$ and $\lim ax_nx = \lim bx_nx = \lim cx_nx = 0$.

A planar space is complete with respect to $(a, b, c)$ iff for every sequence $\{x_i\}$ convergent with respect to $(a, b, c)$, there exists a point $x$ of $P$ with $\lim x_i = x$.

**Theorem.** If $P$ is a convex space which is complete with respect to $(p, a, b)$, then $M_p(a, b)$ is a complete, convex metric space.

If $x$ and $z$ are elements of $M_p(a, b)$, then they are elements of $P$ also. From convexity, there exists a $y$ in $P$ such that $B(x, y, z)$ holds. This gives $xyz = 0$, so that $y$ is in $M_p(a, b)$, and $pxy + pyz = pxz$ which results in $d(x, y) + d(y, z) = d(x, z)$. But, then $y$ is a between point of $x$ and $z$, so that $M_p(a, b)$ is convex. If $\{x_n\}$ is a convergent sequence in $M_p(a, b)$, then $\lim d(x_i, x_j) = 0$. But this implies that $\lim px_i x_j = 0$. Then, since $ax_i x_j = bx_i x_j = 0$ and $pab \neq 0$, $\{x_n\}$ is a convergent sequence with respect to $(p, a, b)$. But $P$ is complete with respect to $(p, a, b)$ so there is an $x$ in $P$ which is the limit of $\{x_n\}$. From $abx = 0$ for every $i$, it follows that $abx = 0$ and that $x$ is in $M_p(a, b)$, which is, therefore, complete.

A subset $S$ of a planar space $P$ is said to be $A$-congruent with a subset $S'$ of a planar space $P'$ (denoted $S \equiv S'$) iff there exists a 1-1 mapping of $S$ onto $S'$ such that $pqr = p'q'r'$, where $p'$, $q'$, $r'$ are the images of $p$, $q$, and $r$.

**Corollary.** If each four points of $P$ are $A$-congruent with four points of $E_3$, then $M_p(a, b)$ is congruently contained in $E_3$.

Let $x$, $y$, $z \in L(a, b)$ and be distinct points. Then $xyz = 0$. If each of the four points of $P$ are $A$-congruently contained in $E_3$, then there exist $p'$, $x'$, $y'$, and $z'$ in $E_3$ with $p'x'y' = pxy$, $p'x'z' = pxz$, $p'y'z' = pyz$ and $x'y'z' = xyz = 0$. Consequently $x'$, $y'$, and $z'$ are collinear and $p'$, $x'$, $y'$, and $z'$ are coplanar. It follows that one of the points $x'$, $y'$, or $z'$ is a between point of the other two. Let $y'$ be the between point. Then $p'x'z' = p'x'y' + p'y'z'$ from which $pxz = pxy + pyz$ follows. But this gives $d(x, z) = d(x, y) + d(y, z)$ so that the three points are embed-
dable in $E_1$. There are more than four distinct points in $M_p(a, b)$ since $a \neq b$ and $M_p(a, b)$ is convex.

Since every semimetric space containing more than four points and having the property that each three of its points are embeddable in $E_1$ is embeddable in $E_1$ [1], $M_p(a, b)$ is congruently contained in $E_1$.

Bibliography

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