ALGEBRAICALLY DEPENDENT FUNCTIONS OF A COMPLEX AND $p$-ADIC VARIABLE

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A set of functions $f_1(z), \cdots, f_m(z)$ are algebraically dependent if there is a polynomial $P(w_1, \cdots, w_m)$ with complex coefficients for which $P(f_1(z), \cdots, f_m(z)) = 0$. In a previous paper (Hilliker [2]) conditions were established under which $m$ functions $f_1(z), \cdots, f_m(z)$ of a complex variable $z$ are algebraically dependent. It was assumed that $f_1(z), \cdots, f_m(z)$ are analytic at one point and that the functional values $f_\mu(z_\nu)$ ($\mu = 1, 2, \cdots, m; \nu = 1, 2, 3, \cdots$) at a convergent sequence of distinct points $\{z_\nu\}$ are algebraic. Let $H(f_\mu(z_\nu))$ denote the height of the algebraic number $f_\mu(z_\nu)$; that is, the maximum of the absolute values of the (integral) coefficients of the minimal polynomial for $f_\mu(z_\nu)$. Let $\deg f_\mu(z_\nu)$ be the degree of $f_\mu(z_\nu)$. Then, roughly speaking, if $H(f_\mu(z_\nu))$ and $\deg f_\mu(z_\nu)$ for $\mu = 1, \cdots, m$ and $\nu = \nu_0 + 1, \nu_0 + 2, \cdots$, are not too large and if the sequence $\{z_\nu\}$ converges sufficiently rapidly, $f_1(z), \cdots, f_m(z)$ are algebraically dependent.

In this paper we wish to relax the restriction on the size of $H(f_\mu(z_\nu))$ and $\deg f_\mu(z_\nu)$ and on the rate of convergence of $\{z_\nu\}$. This is accomplished by assuming that the functions $f_1(z), \cdots, f_m(z)$ are simultaneously analytic in the complex and $p$-adic sense at some point. All of the theorems concerning algebraicness and algebraic dependence in the paper Hilliker [2] lend themselves to $p$-adic generalizations. To illustrate the type of extension that is possible we shall generalize Theorems IV and VII.

Let $Q$ denote the field of rational numbers. We remind the reader of the following concepts. For $a/b \in Q$ we can write, for a given prime $p$, $a/b = p^{-\alpha}a'/b'$ where $a'$ and $b'$ are not divisible by $p$ and where $\alpha$ is an integer. The $p$-adic valuation of $a/b$ is $|a/b|_p = p^{-\alpha}$. The completion of $Q$ under this valuation is the $p$-adic number field $Q_p$ which is analogous to the field of real numbers which is the completion of $Q$ under the ordinary absolute value. The analogue of the field of complex numbers is obtained by taking the algebraic closure of $Q_p$ and then completing this field under the extension of the $p$-adic valuation. We denote this field by $\Omega_p$. It is now possible to consider analytic functions over $\Omega_p$. If a function $f(z)$ is given by a power series with coefficients which are in the complex field and $\Omega_p$, that is, if $f^{(n)}(0)$ is in both fields for $n = 0, 1, \cdots$, then $f(z)$ may be considered to be simultaneously a function of a complex and a $p$-adic variable. For
instance the coefficients could be rational. In fact, it is meaningful to speak of an algebraic number as belonging to both $\Omega_p$ and the complex field. Indeed let $\alpha$ be algebraic. The minimal equation for $\alpha$ may factor over $Q_p$. Let $a_0x^n + \cdots + a_n$ be an irreducible factor over $Q_p$ of the minimal equation where $a_0, \cdots, a_n$ are relatively prime integers in $Q_p$. Then the equation $a_0x^n + \cdots + a_n = 0$ has a solution in $\Omega_p$. One of these solutions is identified with $\alpha$ and the extension of the $p$-adic valuation on $Q$ gives

$$| \alpha |_p = \left| \frac{a_n}{a_0} \right|_p^{1/n}.$$  

Let $| a |_\infty$ denote the ordinary absolute value of the number $a$.

Let

$$f_\mu(z) = \sum_{i=0}^{\infty} a_{i\mu}z^i \quad (\mu = 1, \cdots, m)$$

be $m$ functions of a complex and $p$-adic variable $z$ where each $a_{i\mu}$ is algebraic. Let $\{z_r\}$ be a sequence of distinct algebraic points. Assume that

$$\lim_{r \to \infty} | z_r |_V = 0$$

for $V = \infty, p$. For example we could have $z_r = v^r/p_r$, where $p_r$ denotes the $r$th prime. Suppose that $f_\mu(z_r)$ is the same algebraic number regardless of which field $z_r$ is in. For some index $\nu_0$ let

$$P_r(V) = | z_{r0+q+1} - z_{r0+q} |_V \cdots | z_{r0+q} - z_{r0} |_V$$

and

$$\rho(V) = \lim inf_{r \to \infty} \frac{\log | P_r(V) |_\infty}{\log \nu}.$$  

Then $\rho(V)$ satisfies $1 \leq \rho(V) \leq \infty$ and is a measure of the rate of convergence of the sequence $\{z_r\}$; the more rapidly the sequence converges, the larger $\rho(V)$. Let

$$\gamma_\mu = \lim sup_{r \to \infty} \frac{\log \log 3H(f_\mu(z_r))}{\log \nu}, \quad \prod_{\mu=1}^{m} \deg f_\mu(z_r) \leq h_r$$

with $h_r$ nondecreasing. Let

$$D_r = \sum_{x = r0+1}^{r0+q} h_x, \quad \deg Q(f_1(z_{r0+q}), \cdots, f_m(z_{r0+q})) \leq d_r$$

and
Theorem 1. Under the above conditions the functions $f_1(z), \ldots, f_m(z)$ are algebraically dependent if they have a positive complex and $p$-adic radius of convergence and if

$$\alpha + \frac{1}{m} \sum_{\mu=1}^{m} \gamma_\mu < \max_{V = \infty, p} \rho(V).$$

Remark. Two sequences $z_\nu(\infty)$, $z_\nu(p)$ of possibly transcendental complex and $p$-adic numbers may be used. In which case $\alpha$, $f_\mu(z_\nu(V))$, and $\gamma_\mu$ would depend on $V$, condition (2) is replaced by

$$\max_{V = \infty, p} \left( \alpha(V) + \frac{1}{m} \sum_{\mu=1}^{m} \gamma_\mu(V) \right) < \max_{V = \infty, p} \rho(V),$$

and $h_\chi$ is replaced by $\max_V h_\chi(V)$.

Proof. The proof is somewhat similar to arguments in the paper [2] so we shall only outline the details.

Let

$$\phi(z) = \sum_{\tau_\mu = 0}^{t_\mu} c_{\tau_1, \ldots, \tau_m} f_1^{\tau_1}(z) \cdots f_m^{\tau_m}(z), \quad (\mu = 1, \ldots, m).$$

We shall show that for suitably chosen integers $t_\mu$ there exists integers $c_{\tau_1, \ldots, \tau_m}$ not all zero and not too large so that $\phi(z) \equiv 0$. Consider the $n$ equations

$$\phi(z) = 0, \quad z = z_{\tau_0+1}, \ldots, z_{\tau_0+n},$$

in $c_{\tau_1, \ldots, \tau_m}$. Let $\beta_\mu = \beta_\mu(v) = f_\mu(z_v)$, $\delta(\mu) = \delta(\mu, \nu) = \deg \beta_\mu$ and $b_{\delta(\mu)} = b_{\delta(\mu)}(v)$ be the denominator of $\beta_\mu$ for $\mu = 1, \ldots, m$. Then the coefficients in (4) can be expressed as

$$\beta_1^{\tau_1} \cdots \beta_m^{\tau_m} = b_{\delta(1)}^{\tau_1} \cdots b_{\delta(m)}^{\tau_m}$$

$$\cdot \sum_{\sigma_\mu = 0}^{\delta(\mu)-1} a(\tau_1, \ldots, \tau_m, \sigma_1, \ldots, \sigma_m) \beta_1^{\sigma_1} \cdots \beta_m^{\sigma_m}$$

where the $a$'s are integers which satisfy

$$| a(\tau_1, \ldots, \tau_m, \sigma_1, \ldots, \sigma_m) |_\infty \leq \prod_{\mu=1}^{m} (2H(\beta_\mu))^\tau_\mu.$$
for $\nu=v_0+1, \ldots, v_0+n$; $\sigma_\mu=0, \ldots, \delta(\mu)-1; \mu=1, \ldots, m$. Thus we have a system (6) of at most $D_n$ equations with integral coefficients in the $\prod_{\mu=1}^{m}(t_\mu+1)$ unknowns $c_{\tau_1} \ldots c_{\tau_m}$. Let

$$t_\mu = \left[ \frac{(2D_n)^{1/m}(v_0 + n)^{(1/m)} \gamma^{\mu-\gamma}}{2} \right]$$

where $[x]$ denotes the greatest integer in $x$. Then the number of unknowns is at least twice the number of equations. The coefficients in (6) have, for a given $\varepsilon > 0$, absolute value at most

$$\prod_{\mu=1}^{m} \{ 2H(f_\mu(z_\nu)) \} t_\mu \leq \exp(c_1 D_n^{1/m} (1/m) \gamma^{\mu-\gamma})$$

where $c_1$ and the $c_i$'s to follow are constants. Thus (6) has a solution in integers $c_{\tau_1} \ldots c_{\tau_m}$, not all zero, for which

$$|c_{\tau_1} \ldots c_{\tau_m}|_\infty \leq \exp(c_2 D_n^{1/m} (1/m) \gamma^{\mu-\gamma}).$$

Hence (4) holds. We shall now show that if $\phi(z) = 0$ for $z=z_{q+1}, \ldots, z_{q+n}$, then $\phi(z) = 0$ for $z=z_{q+n+1}$ if $n_1 \geq n$ and $n$ is sufficiently large. By applying the maximum principle to the analytic function

$$\phi(z) \frac{1}{(z - z_{q+1}) \cdots (z - z_{q+n})}$$

in some small closed disk about the origin we obtain an upper bound for $|\phi(z)|_{V}$. An upper bound for the conjugates can be obtained directly from (3). Let $K$ be the denominator of $\phi(z_{q+n+1})$. Then if we combine these estimates we conclude that the norm of the algebraic integer $K\phi(z_{q+n+1})$ satisfies

$$\prod_{V=\infty, p} N(K\phi(z_{q+n+1})) \leq \exp \left(c_3 n_1^{1/(m)} \gamma^{\mu+2\varepsilon} + c_4 n_1 - \sum_{V} n_1^{(V)-1/2} \log n_1 \right).$$

If $\rho(V) \neq 1$ for some $V$ we see by (2) that the right side of (7) is less than 1. Thus $\phi(z_{q+n+1}) = 0$ for otherwise the left side of (7) would be at least 1. Thus $\phi(z)$ vanishes at the entire sequence $z_{q+1}, z_{q+2}, \ldots$ and hence it must be identically zero. If $\rho(V) = 1$ for each $V$, then the term
may be replaced by $c_5 n_1$ with $c_5 > c_4$ and again $\phi(z) = 0$.

If equality holds in (2), then more complicated conditions are required. Here it becomes significant to consider, say, $r$ primes $p_1, \ldots, p_r$. Let $f_\mu(z), \mu = 1, \ldots, m$, be given by (1) with $a_{i,\mu}$ algebraic. Let $z_\mu(V)$ be a sequence of distinct points, not necessarily algebraic, for $V = \infty, p_1, \ldots, p_r$ with $z_\mu(\infty)$ complex and $z_\mu(p_i)$ in $\Omega_{p_i}, i = 1, \ldots, r$. Suppose that $\lim_{\nu \to \infty} |z_\mu(V)|_\nu = 0$ for $V = \infty, p_1, \ldots, p_r$ and that $f_\mu(z_\mu(V))$ is algebraic for $\mu = 1, \ldots, m; \nu = 1, 2, \ldots; V = \infty, p_1, \ldots, p_r$. For simplicity in the statement of the theorem, we assume that the degree of $f_\mu(z_\mu(V))$ is bounded. Here $f_\mu(z_\mu(V))$ and $\gamma_\mu$ depend on $V$. Let

$$\Omega_\mu(V) = \limsup_{\nu \to \infty} \frac{\log H(f_\mu(z_\mu(V)))}{\nu^\gamma_\mu(V) \log \nu}$$

and

$$\sigma(V) = \liminf_{\nu \to \infty} \frac{\log P_\nu(V)}{\nu^\rho(V) \log \nu}.$$ 

Then $\sigma(V)$ is a refinement of the measure $\rho(V)$ of the sequence $\{z_\nu(V)\}$. Let

$$\deg Q(f_1(z_\nu(V)), \ldots, f_m(z_\nu(V))) \leq d$$

for $\nu = 1, 2, \ldots$ and $V = \infty, p_1, \ldots, p_r$.

**Theorem 2.** Under the above conditions the functions $f_1(z), \ldots, f_m(z)$ are algebraically dependent if they have a positive complex and $p_i$-adic radius of convergence for $i = 1, \ldots, r$ and if either (a) or (b) below hold.

(a) $\max_V (1/m + (1/m)^2 \gamma_\mu(V)) < \max_V \rho(V)$.

(b) $1/m + (1/m)^2 \gamma_\mu(V) = \rho(V)$ for each $V, \rho(\infty) = \rho(p_1) = \cdots = \rho(p_r) = \gamma_\mu$, for each $\mu$, $\Omega_\mu(V) < \infty$, for each $V$ and $\mu$ and

$$4md^2 \sum_V \left(2(r + 1) \prod_{\mu=1}^m \Omega_\mu(V)\right)^{1/m} < \sum_V \sigma(V).$$

The proof is similar to that of Theorem 1.

**References**
