ON $\mathfrak{F}$-NORMALIZERS AND $\mathfrak{F}$-COVERING SUBGROUPS

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Recently, Carter and Hawkes [2] have generalized the construction of system normalizers of finite solvable groups, introducing the concept of an $\mathfrak{F}$-normalizer, where $\mathfrak{F}$ is any local formation. In this note we give a different proof of one of their main results, namely, each $\mathfrak{F}$-normalizer is contained in an $\mathfrak{F}$-covering subgroup. Our proof avoids the study of the imbedding of $\mathfrak{F}$-normalizers in maximal subgroups, and is similar to Huppert's proof that each system normalizer is contained in a Carter subgroup [4, p. 37].

We first recall the definitions. All groups in this note are solvable and finite. A formation of groups is a class of groups closed under homomorphic images and subdirect products. If formations $\mathfrak{F}(p)$, one for each prime $p$, are given, the local formation $\mathfrak{F}$ locally defined by $\{\mathfrak{F}(p)\}$ is the class of all groups $G$ such that, whenever $M$ is a chief factor of $G$, of order $p^n$, say, then the automorphism group induced on $M$ by $G$ belongs to $\mathfrak{F}(p)$. (We are assuming that $\mathfrak{F}(p) \neq \emptyset$ for each $p$.)

Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}(p)\}$. For each $p$, let $N_p$ be the unique minimal normal subgroup of $G$ such that $G/N_p \in \mathfrak{F}(p)$. Let $T^p$ be a $p$-complement of $N_p$. Then $D = \bigcap_p N(T^p)$ is an $\{\mathfrak{F}(p)\}$-normalizer of $G$. If $\mathfrak{F}(p) \subseteq \mathfrak{F}$, for each $p$, then the $\{\mathfrak{F}(p)\}$-normalizers depend only on $\mathfrak{F}$, and not on $\mathfrak{F}(p)$, and are called $\mathfrak{F}$-normalizers. An $\mathfrak{F}$-covering subgroup of $G$ is a subgroup $C$ such that $C \subseteq \mathfrak{F}$ and, whenever $C \subseteq K \subseteq G$, $L \triangleleft K$ and $K/L \in \mathfrak{F}$, then $K = LC$. If $\mathfrak{F}$ is local, then $\mathfrak{F}$-covering subgroups of $G$ exist, and are unique up to conjugacy [3].

We use below the (known) fact that a homomorphism maps $\mathfrak{F}$-normalizers onto $\mathfrak{F}$-normalizers, and $\mathfrak{F}$-covering subgroups onto $\mathfrak{F}$-covering subgroups.

**Theorem.** Let $\mathfrak{F}$ be locally defined by $\{\mathfrak{F}(p)\}$. If, for each $p$, either $\mathfrak{F}(p) \subseteq \mathfrak{F}$ or $\mathfrak{F}(p)$ is subgroup closed, then each $\mathfrak{F}$-covering subgroup of $G$ contains an $\{\mathfrak{F}(p)\}$-normalizer of $G$.

**Proof.** Let $C$ be an $\mathfrak{F}$-covering subgroup of $G$, and let $N_p$ and $T^p$ be defined as above. We say that the system $\{T^p\}$ reduces to $C$, if there exist $p$-complements of $G$, $S^p$, such that $T^p = S^p \cap N_p$, and $S^p \cap C$ is a $p$-complement of $C$. Given a system $\{C^p\}$ of $p$-complements of $C$, one can choose $p$-complements $\{S^p\}$ of $G$ such that $C^p \subseteq S^p$. De-
noting now $T^p = S^p \cap N_p$, the system $\{ T^p \}$ reduces to $C$. Hence such systems exist.

We shall prove, by induction on $|G|$, that if $\{ T^p \}$ reduces to $C$, then $D = \bigcap_p N(T^p)$ is contained in $C$.

If $G = C$, there is nothing to prove, so we assume $G \neq C$, so that $G \in \mathfrak{H}$. Let $M$ be a minimal normal subgroup of $G$. By induction, $D_M/M \subseteq C_M/M$, hence $D \subseteq C_M$.

Suppose first that $CM \neq G$. Let $\{ S^p \}$ be the $p$-complements of $G$ defined above. Since $\{ S^p \}$ reduces to $C$ and $M \triangleleft G$, $\{ S^p \}$ reduces to $CM$ [1, Corollary 2.8]. Denote $U^p = S^p \cap CM$. Let $Q_p$ be the minimal normal subgroup of $CM$, for which $CM/Q_p \in \mathfrak{H}(p)$. If $\mathfrak{H}(p) \subseteq \mathfrak{H}$, then the defining properties of $C$ imply $G = CN_p$; therefore $CM/CM \cap N_p \cong G/N_p \subseteq F(p)$. If $\mathfrak{H}(p)$ is subgroup closed, the same conclusion is true, since $CM/CM \cap N_p$ is isomorphic to a subgroup of $G/N_p$. Hence in any case $Q_p \subseteq CM \cap N_p$.

Now the system $\{ T^p \cap Q_p \} = \{ S^p \cap Q_p \} = \{ U^p \cap Q_p \}$ reduces to $C$, which is an $\mathfrak{H}$-covering subgroup of $CM$. By induction, $\bigcap N_{CM}(T^p \cap Q_p)$ is contained in $C$. Obviously, $D \subseteq \bigcap N(T^p \cap Q_p)$ for all $p$, so $D \subseteq C$.

Now assume $G = CM$. Let $|M| = q^n$, for some prime $q$. By assumption, $G \in \mathfrak{H}$, but $G/M \cong C \in \mathfrak{H}$. Hence, by Gaschütz's construction of $\mathfrak{H}$-covering subgroups [3], $C = N(V^q)$, where $V^q$ is a $q$-complement of $O_{q'}(G \bmod M)$. Since $|G:C| = q^n$, and $S^q \cap C$ is a $q$-complement of $C$, $S^q \subseteq C$, so also $T^q \subseteq C$. As $G/M \in \mathfrak{H}$, we must have $N_q \subseteq O_{q'q}(G \bmod M)$; therefore $T^q \subseteq O_{q'}(G \bmod M)$. As $T^q \subseteq C$, $T^q$ normalizes $V^q$; therefore $T^q \subseteq V^q$, $T^q = V^q \cap N_q$. This implies $T^q < C$. Since $C$ is maximal, $N(T^q) = C$ or $N(T^q) = G$. In the second case $N_q$ has a normal $q$-complement; hence $N_q \subseteq O_{q'q}(G)$, which is equivalent to $G$ inducing on all chief factors of orders $q^n$ a group belonging to $\mathfrak{H}(p)$. A chief factor of order $p^m$, $p \neq q$, is operator isomorphic to one of $G/M$; hence $G$ certainly induces on it a group belonging to $\mathfrak{H}(p)$. Therefore $G \in \mathfrak{H}$, a contradiction. Hence $C = N(T^q)$ and $D \subseteq C$.

REFERENCES


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