FUNCTIONS OF BOUNDED VARIATION AND MOMENT-SEQUENCES OF CONTINUOUS FUNCTIONS

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This paper gives a necessary and sufficient condition for certain functions to be of bounded variation.

Let $U_0$ denote (see [1, p. 24]) the class of functions from $[0, 1]$ to the real numbers to which $f$ belongs only in case $f(0^+)$ exists, $f(1^-)$ exists, and, for each number $x$ between 0 and 1, $f(x^-)$ exists, $f(x^+)$ exists, and either $f(x^-) \leq f(x) \leq f(x^+)$ or $f(x^+) \leq f(x) \leq f(x^-)$.

Theorem. If $f$ is in $U_0$, then a necessary and sufficient condition that $f$ be of bounded variation is that, for each moment-sequence $c$ of a continuous function from $[0, 1]$ to the real numbers, the limit

$$M_f(c) = \lim_{n \to \infty} \sum_{k=0}^{n} f(k/n) \left(\begin{array}{c} n \\ k \end{array}\right) \Delta^{n-k} c_k$$

exists. Furthermore, if $f$ is of bounded variation and $c$ is the moment-sequence of the continuous function $g$ from $[0, 1]$ to the real numbers, then $M_f(c) = \int_0^1 f \, dg$.

This theorem bears some similarity to a theorem of MacNerney [2, p. 368] which states, in terms of the kind of limit described above, a necessary and sufficient condition for a sequence to be the moment-sequence of a function of bounded variation. Also, Tonne [3] has used this kind of limit to “integrate” certain functions with respect to certain sequences in the sense that one might describe $M_f(c)$ as the integral of $f$ with respect to $c$.

Definitions. If each of $n$ and $k$ is a nonnegative integer and $c$ is a number-sequence and $f$ is a function from $[0, 1]$ to the real numbers, then

$$\Delta^n c_k = \sum_{q=0}^{n} (-1)^q \left(\begin{array}{c} n \\ q \end{array}\right) c_{k+q},$$

$$L(f, c)_n = \sum_{p=0}^{n} f(p/n) \left(\begin{array}{c} n \\ p \end{array}\right) \Delta^{n-p} c_p,$$

$I$ is the identity function on the interval $[0, 1]$, $B(f)_n$ is the Bernstein polynomial for $f$ of order $n$, namely,

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\[ \sum_{p=0}^{n} f(p/n) \binom{n}{p} (1 - I)^{n-p} I^p, \]

\[ \int_0^1 |df| \] denotes the total variation of \( f \) on \([0, 1]\) if \( f \) is of bounded variation, and \( c \) is the moment-sequence of \( f \): \( c_p = \int_0^1 I^p df \) (\( p = 0, 1, \cdots \)).

**Proof of Theorem.** Suppose that \( f \) is of bounded variation and \( c \) is the moment sequence of the continuous function \( g \) from \([0, 1]\) to the real numbers. If \( n \) is a positive integer,

\[ L(f, c)_n = \sum_{p=0}^{n} f(p/n) \binom{n}{p} \Delta^{n-p} c_p \]

\[ = \sum_{p=0}^{n} f(p/n) \binom{n}{p} \int_0^1 (1 - I)^{n-p} I^p dg \]

\[ = \int_0^1 B(f)_n dg \]

\[ = -\int_0^1 g dB(f)_n + B(f)_n(1) \cdot g(1) - B(f)_n(0) \cdot g(0) \]

\[ = -\int_0^1 g dB(f)_n + f(1)g(1) - f(0)g(0). \]

Since \( f \) is in \( U_0 \), the Bernstein polynomial sequence \( B(f) \) has limit \( f \) on \([0, 1]\) except, perhaps, at countably many points of \([0, 1]\) (see [1, p. 27]). The sequence \( B(f) \) is "uniformly of bounded variation" (see [1, p. 25]). So (see [4, p. 31]) the limit of the sequence \( L(f, c) \) is

\[ -\int_0^1 g df + f(1)g(1) - f(0)g(0), \]

which is \( \int_0^1 f dg \).

On the other hand, suppose that, for each moment sequence \( c \) of a continuous function \( g \) from \([0, 1]\) to the real numbers, the limit \( M_f(c) \) exists. Suppose that \( g \) is a continuous function from \([0, 1]\) to the numbers and let \( c \) be its moment-sequence and, for each positive integer \( n \), let \( T_n(g) \) be \( L(f, c)_n - f(1)g(1) + f(0)g(0) \). With the aid of the preceding computation for \( L(f, c)_n \) we see that \( T_n \) is a continuous linear transformation from the space of continuous functions on \([0, 1]\) (with maximum modulus norm) to the real numbers; the norm of \( T_n \) is \( \int_0^1 |dB(f)_n| \). For each continuous function \( g \) from \([0, 1]\) to the numbers, the sequence \( T(g) \) converges, so that, by the "principle of uniform boundedness," there is a number \( K \) such that if \( n \) is a positive integer then \( \int_0^1 |dB(f)_n| < K \), so that (see [1, p. 25]) \( f \) is of bounded variation on \([0, 1]\).
References


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