A NOTE ON FIXED POINT THEOREMS FOR A FAMILY OF NONEXPANSIVE MAPPINGS

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1. Introduction. In this note we prove the two following theorems:

THEOREM 1. Let $X$ be a Banach space and $K$ a nonempty closed convex subset of $X$. Let $\mathcal{F}$ be a commuting family of nonexpansive mappings from $K$ into itself and $M$ a compact subset of $X$ such that there exist an $f_1 \in \mathcal{F}$ and an $x_0 \in K$ satisfying the following properties:

(i) $\{f_n^1(x_0)\}$ is a bounded set,
(ii) $\overline{\text{cl}} \{f_n^1(x_0)\} \cap M \neq \emptyset$ for every $x \in K$. Then the family $\mathcal{F}$ has a common fixed point in $M$.

Theorem 1 is a generalization of Theorem 1 in [1] of L. P. Belluce and W. A. Kirk where $K$ is a bounded set. Similarly, Theorem 2 is a generalization of Theorem 2 in [2] of F. E. Browder.

2. Definition and notations. Let $X$ be a Banach space. A mapping $f$ from a subset $A$ of $X$ into itself is nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$, for every $x, y \in A$. $f^n(x)$ is defined inductively as $f[f^{n-1}(x)]$, and hence $\{f^n(x_0)\}$ the set of iterate images of $x_0$. We denote the diameter of a set $A$ by $d(A)$, the closure and the closure convex by $\overline{\text{cl}}(A)$ and $\overline{\text{co}}(A)$ respectively.

The proof of Theorem 1 is in the general line of argument of L. P. Belluce and W. A. Kirk in [1]. Theorem 2 can be seen as a corollary of Theorem 2 in [2].

Proof of Theorem 1. Suppose that the set $\{f_n^1(x_0)\}$ be bounded by the number $d$. Let $B_n$ denote the closed ball of center $f_n^1(x_0)$ and radius $d$. We define: $D_k = \bigcap_{n=k}^\infty (B_n \cap K)$ and $D = \overline{\text{cl}}(\bigcup_{k=1}^\infty D_k)$.

Then one can show that $D$ is a nonempty closed and bounded convex set which is mapped into itself by the mapping $f_1$. Applying
Theorem 1 in [1] to the case where $\mathcal{F} = \{f_1\}$, we can get a fixed point of $f_1$ in $M$. The condition (ii) implies that every fixed point of $f_1$ must be in $M$. Hence, the set $H_1$ of all fixed points of $f_1$ is a nonempty closed compact subset of $M$. Furthermore, by the commutativity property of the family $\mathcal{F}$, $f(H_1) \subset H_1$ for every $f \in \mathcal{F}$. Also, by compactness of $H_1$ and by Zorn’s lemma, there is a set $H^+$ which is minimal with respect to being nonempty, compact subset of $H_1$ and mapped into itself by every $f \in \mathcal{F}$. Since for every $f, g \in \mathcal{F}$ we have

$$g[f(H^+)] = f[g(H^+)] \subset f(H^+);$$

therefore $f(H^+)$ is a nonempty compact subset of $H_1$ and mapped into itself by each $g \in \mathcal{F}$. Thus, by minimality of $H^+$, $f(H^+) = H^+$ for every $f \in \mathcal{F}$. Let $C$ be the set defined as follows:

$$C = \{x \in K | \|x - y\| \leq d(H^+) \text{ for every } y \in H^+\}.$$ 

Since $H^+$ is a nonempty set, $C$ is a nonempty closed bounded convex subset of $K$. Furthermore, $f(H^+) = H^+$ implies that $f(C) \subset C$, for every $f \in \mathcal{F}$. As a consequence of Theorem 1 in [1], the family $\mathcal{F}$ has a common fixed point in $C$. By the condition (ii), this common fixed point must lie in $M$.

**Proof of Theorem 2.** By the same argument as in the proof of Theorem 1, the set $H_1$ of all fixed points of $f_1$ is a nonempty closed subset of $M$ and hence, also a bounded set. Furthermore, by the uniform convexity of the space $X$, the set $H_1$ is a convex set. Also, the commutativity property of the family $\mathcal{F}$ implies that $f(H_1) \subset H_1$ for every $f \in \mathcal{F}$. As a consequence of Theorem 2 in [2], the family $\mathcal{F}$ has a common fixed point in $H_1$ and hence in $M$.

**References**