ON SUSPENDING HOMOTOPY SPHERES

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A homotopy n-sphere is a compact topological n-manifold having the homotopy type of $S^n$ and a fake n-cell a compact, contractible, topological n-manifold whose boundary is homeomorphic to $S^{n-1}$. The object of this paper is to establish the following two propositions as regards such classes of manifolds.

THEOREM 1. The suspension of a homotopy 4-sphere is homeomorphic to $S^5$.

THEOREM 2. The suspension of a fake 4-cell is homeomorphic to $I^5$.

It should be noted that M. Hirsch [4] has proved both theorems for the case in which the manifolds are smooth by the methods of differential topology. The case for piecewise linear (combinatorial) manifolds follows immediately as a result of their admitting compatible smooth structures.

We prove Theorem 2 first as follows: Let $F^4$ be a fake 4-cell and $h: Bd F^4 \times [0, 1) \to F^4$ a collar for $Bd F^4$ in $F^4$. Put $X = Cl(F^4 - h[0, 1/2])$. Note that $X$ is homeomorphic to $F^4$. Let $N^5 = F^4 \times [-2, 2]$. We claim that $Int N^5$ is a contractible open 5-manifold which is 1-connected at infinity, admits a piecewise linear triangulation and thus by Stallings [6] is homeomorphic to $E^5$. Obviously the interior of $N^5$ is contractible and since by Van Kampen's theorem $Bd N^5$ is simply connected (it is in fact a homotopy 4-sphere) $Int N^5$ is 1-connected at infinity. It is easily seen that the double 2$N^5$ of $N^5$ is a homotopy 5-sphere and thus by Poincaré's conjecture for topological n-manifolds ($n \geq 5$), which was proved by E. H. Connell and can be found in M. H. A. Newman [5], a 5-sphere. Hence $Int N^5$ is homeomorphic to an open subset of $S^5$. Consequently, it admits a piecewise-linear triangulation and is homeomorphic to $E^5$. Clearly $X \times (-1)$ and $X \times 1$ are cellular in $Int N^5$, for they have arbitrarily small neighborhoods homeomorphic to $Int N^5$. It follows (Theorem 1 of [1]) that there is a mapping $F: N^5 \to N^5$ such that $F|Bd N^5 = id$ and the only nondegenerate inverse sets of $F$ are $X \times (-1)$ and $X \times 1$. Let $g: Bd X \times [-1, 1] \to S(Bd X)$ be the natural mapping. $Fg^{-1}$ is a homeomorphism of $S(Bd X)$ onto $F(Bd X \times [-1, 1])$. Hence $F(Bd X \times [-1, 1])$ is a 4-sphere. Clearly $F(Bd X \times [-1, 1])$ is locally flat in $Int N^5$ except

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possibly at \( F(Bd X \times -1) \) and \( F(Bd X \times 1) \). Hence by Chernavskii [3] (or by results submitted to the Annals of Mathematics by R. C. Kirby) locally flat. By the Generalized Schoenflies Theorem \( F(X \times [-1, 1]) \) is a cell. If \( G: X \times [-1, 1] \rightarrow S(X) \) is the natural map (which extends \( g \)), then \( FG^{-1} \) is a homeomorphism of \( S(X) \) onto the 5-cell \( F(X \times [-1, 1]) \). Since \( X \) is homeomorphic to \( F^4 \), \( S(F^4) \) is a 5-cell.

Theorem 1 now follows by removing from a homotopy 4-sphere \( M^4 \) the interior of a 4-simplex (4-cell with locally flat boundary) \( \sigma^4 \) to obtain a fake 4-cell \( F^4 \). By Theorem 1, \( S(F^4) \) is a cell. Hence \( S(M^4) = S(F^4) \cup S(\sigma^4) \) where \( S(F^4) \cap S(\sigma^4) = S(Bd \sigma^4) \) and is thus a 5-sphere.

Theorem 2 also follows from Theorem 1 by reversing the argument above and using the following theorem of J. C. Cantrell [2].

**Theorem 3.** If the \((n-1)\)-sphere \((n > 3)\) \( S \) in \( S^n \) is locally collared at a point \( p \) in one side and locally collared in a deleted neighborhood of \( p \) on the other side, then \( S \) is locally flat at \( p \).

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**Bibliography**


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