LINEAR SYSTEMS OF DIFFERENCE EQUATIONS
WITH A REGULAR SINGULARITY

W. A. HARRIS, JR.¹

1. Introduction. This paper is concerned with the linear system of difference equations

(1.1) \( w(z + 1) = A(z)w(z) \)

where \( w \) is a vector with \( n \) components and \( A \) is an \( n \) by \( n \) matrix which admits the generalized factorial series representation

(1.2) \( A(z) = z^n \sum_{k=0}^{\infty} A_k z^{-[k]}, \quad \text{Re}\{z\} > \mu, \)

where \( z^{-[k]} = \{z(z+1) \cdots (z+k-1)\}^{-1} \) and \( z^{[0]} = 1 \).

In an analogous manner to linear systems of differential equations with singularities (at \( z = \infty \)) we make the following definitions [1, p. 73], [5, p. 111].

The system \( w(z + 1) = A(z)w(z) \) is said to have a singularity of the first kind if \( A(z) \) admits the factorial series representation

(1.3) \( A(z) = I + \sum_{k=1}^{\infty} A_k z^{-[k]}, \quad \text{Re}\{z\} > \mu, \)

and otherwise a singularity of the second kind.

The system \( w(z + 1) = A(z)w(z) \) is said to have a regular singularity if there exists a fundamental matrix of the form

(1.4) \( W(z) = S(z)z^R \)

such that \( S(z) \) admits a generalized factorial series representation

(1.5) \( S(z) = z^p \sum_{k=0}^{\infty} S_k z^{-[k]}, \quad \text{Re}\{z\} > \mu', \)

and \( R \) is a constant matrix.

Linear systems of difference equations with a singularity of the first kind have been extensively studied by the author [2] and such systems are known to have a regular singularity. However, the converse is not true. Indeed, a necessary and sufficient condition that a linear system of difference equations \( w(z + 1) = A(z)w(z) \) have a regular

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singularity is that there exists a nonsingular matrix \( T(z) \) which admits a generalized factorial series representation such that the transformation \( u = Tu \) yields a system \( u(z+1) = B(z) u(z) \) which has a singularity of the first kind, i.e.

\[
B(z) = T^{-1}(z+1) A(z) T(z) = I + \sum_{k=1}^{\infty} B_k z^{-k}.
\]

If \( A, B \) and \( T \) admit generalized factorial series representations and \( \det T(z) \neq 0 \), we shall denote the equivalence relation \( B(z) = T^{-1}(z+1) A(z) T(z) \) by \( A \sim B \).

Even though this condition is necessary and sufficient for the desired structure of a fundamental matrix, it cannot be used to resolve the question for a preassigned system. However, the author \[3\]^2 has given an algorithm to determine whether a given system has a regular singularity which is contained in the following theorems.

**Theorem (Harris).** Let \( A(z) \) admit a generalized factorial series representation, \( A(z) = z^p \sum_{k=0}^{\infty} A_k z^{-k}, A_0 \neq 0, \Re \{z\} > a \). A necessary condition that the system \( w(z+1) = A(z) w(z) \) has a regular singularity is that \( p \geq 0 \), and \( A_0 \) or \( A_0 - I \) be nilpotent for \( p > 0 \) or \( p = 0 \) respectively.

**Theorem (Harris).** Let \( A(z) \) and \( B(z) \) admit factorial series representations, \( A(z) = \sum_{k=0}^{\infty} A_k z^{-k}, B(z) = \sum_{k=0}^{\infty} B_k z^{-k} \), \( \Re \{z\} > a \), and let \( A_0 \neq 0 \). A necessary and sufficient condition that \( A \sim B \) such that \( r = \text{rank}(A_0 - pI) > \text{rank}(B_0 - pI) \) for some \( p \) is that the polynomial

\[
\mathfrak{B}(\lambda) = \{ z^{-r} \det[\lambda I + z(A(z) - \rho I)] \} \big|_{z=\infty} = \sum_{k=0}^{n-r} \lambda^k \mathfrak{B}_k(A_0, A_1)
\]

vanish identically in \( \lambda \).

In this paper we shall derive necessary conditions for a given system \( w(z+1) = A(z) w(z) \) to have a regular singularity based on the characteristic polynomial of the matrix \( A(z) \). These results parallel recent results of D. A. Lutz \[6\] for linear differential systems with a regular singular point.

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2. **Statement of results.** It is natural and convenient to associate with the matrix \( A(z) \) the matrix \( \tilde{A}(z) \) defined by

\[
(2.1) \quad \tilde{A}(z) = A(z) - I.
\]

^2 This paper assumes that \( T^{-1}(z) \) also admits a generalized factorial series representation; a fact subsequently proved by the author and H. L. Turrittin \[4\].
**Definition.** The symmetric function of rank $k$ of a matrix $A$ is the coefficient of $\lambda^{n-k}$ in the polynomial

$$U(\lambda) = \det(\lambda I + A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$ 

For a system $u(z+1) = B(z)u(z)$ with a singularity of the first kind we have: the symmetric function of rank $k$ of the matrix $B(z)$ satisfies the order condition $b_k(z) = O(z^{-k}).$

Since, for every system $w(z+1) = A(z)w(z)$ with a regular singularity, we have $A \sim B$ where the system $u(z+1) = B(z)u(z)$ has a singularity of the first kind, i.e. $A(z) = T(z+1)B(z)T^{-1}(z),$ it is natural to expect that the orders of the $n$ symmetric functions of $A$ are somewhat restricted. This is indeed the case and we state:

**Theorem 1.** If the system $w(z+1) = A(z)w(z)$ with

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-k}, \quad A_0 \neq 0, \quad \Re\{z\} > \mu$$

has a regular singularity, then the symmetric functions of $A$ satisfy

$$a_n(z) = O(z^{(n-1)q-2} + z^{-n}), \quad a_k(z) = O(z^{q-2} + z^{-k}), \quad k = 1, \ldots, n - 1.$$ 

We shall show by example that this result is sharp for all $k, n$ and $q \geq 0.$

In the same manner we can also state

**Theorem 2.** If the system $w(z+1) = A(z)w(z)$ with

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-k}, \quad A_0 \neq 0, \quad \Re\{z\} > \mu$$

has a regular singularity and we write

$$A(z) = A(z) - I = z^q \sum_{k=0}^{\infty} A_k z^{-k},$$

then

(i) $\hat{A}_0 = 0$ for some $k \leq n,$ i.e. $\hat{A}_0$ is nilpotent and

(ii) $\text{trace } (\hat{A}_k \hat{A}_1) = 0$ for $k = 0, 1, \ldots, n - 1$ in case $q \geq 1$ and for $k = 1, 2, \ldots, n - 1$ in case $q = 0.$

Even though the order conditions for the symmetric functions given in Theorem 1 are sharp, they are clearly not sufficient for a regular singularity. However, we do have the following partial converse to Theorem 1.
Theorem 3. If
\[ A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-k}, \quad A_0 \neq 0, \; q \geq 0, \; \text{Re}\{z\} > \mu \]
and the symmetric functions of \( A(z) \) satisfy \( a_k(z) = O(z^{q-2}+z^{-k}), \; k = 1, 2, \ldots, n, \) then

(i) \( A_0 = 0, \)
and if \( A_0^{-1} \neq 0, \) then

(ii) \( \text{trace} (A_0 A_1) = 0, \; k = 0, \ldots, n-1 \) for \( q \geq 1 \) and \( k = 1, \ldots, n-1 \) for \( q = 0. \)

Note that if \( n \geq 2, \) the terms \( z^{-n} \) and \( z^{-k} \) may be dropped in the order conditions except for \( k = 1 \) when \( q = 0. \)

We also have the following characterization of the order conditions.

Corollary. Let \( A_0^{-1} \neq 0. \) Then the symmetric functions of \( A(z) \) satisfy the order conditions \( a_k = O(z^{q-2}) \) if and only if there exists a transformation matrix \( P(z) = I + P_1 z^{-1} \) such that

\[ P^{-1}(z+1)A(z)P(z) = z^q A_0 + O(z^{q-2}), \quad q \neq 1 \]
\[ = z A_0 + I + O(z^{-1}), \quad q = 1. \]

3. Preliminary lemma. A useful tool for the proof of these theorems is the following lemma which is a generalization to matrices of factorial series of the fact that an analytic function \( f(z) \neq 0 \) with at most a pole at \( z = \infty \) can be written in the form \( f(z) = z^a g(z) \) where \( g(z) \) is analytic at \( z = \infty \) and \( g(\infty) \neq 0. \)

Lemma. Let \( T(z) \) admit a factorial series representation

\[ T(z) = \sum_{k=0}^{\infty} T_k z^{-k}, \quad T_0 \neq 0, \; \text{Re}\{z\} > a. \]

Then \( T(z) \) can be represented in the form

\[ T(z) = P(z) z^{-D} Q(z) \]

where \( P(z) \) is a polynomial in \( z^{-1}, \; \det P(z) = 1, \) \( Q(z) \) admits a factorial series representation

\[ Q(z) = \sum_{k=0}^{\infty} Q_k z^{-[k]}, \quad \det Q_0 \neq 0, \]

and

\[ z^{-D} = \text{diag}(z^{-d_1}, z^{-d_2}, \ldots, z^{-d_n}) \]

where \( 0 = d_1 \leq d_2 \leq \cdots \leq d_n \) are integers.
For a proof of this Lemma, see Harris [3, p. 257].

4. Proof of Theorem 1. If the system \( w(z+1) = A(z)w(z) \) has a regular singularity, there exists a fundamental matrix of the form \( W(z) = S(z)z^R \) and without loss of generality we may assume that

\[
S(z) = \sum_{k=0}^{\infty} S_k z^{-[k]}, \quad S_0 \neq 0, \ Re\{z\} > a.
\]

Hence \( S(z) \) has the representation given in the preceding Lemma,

\[
S(z) = P(z)z^{-D}Q(z).
\]

Thus

\[
A(z) \sim B(z) = P^{-1}(z + 1)A(z)P(z), \\
B(z) \sim C(z) = (z + 1)^DB(z)z^{-D}, \\
C(z) \sim H(z) = Q^{-1}(z + 1)C(z)Q(z) = I + O(z^{-1}).
\]

Since \( Q_0 \) is nonsingular, \( C(z) = I + O(z^{-1}) \). Writing

\[
B(z) = (z + 1)^{-D}C(z)z^D = z^{-D}(1 + z^{-1})^{-D}C(z)z^D
\]

and noting that \((1 + z^{-1})^{-D} = I + O(z^{-1})\), we have that \( \det(\lambda I + \hat{B}) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_n \), where

\[
(4.1) \quad b_k(z) = O(z^{-k}), \quad k = 1, \ldots, n.
\]

We have

\[
A(z) = P(z)\left\{ B(z) + (P^{-1}(z)P(z + 1) - I)B(z) \right\} P^{-1}(z)
\]

\[
= P(z)\left\{ B(z) + F(z) \right\} P^{-1}(z).
\]

Since \( P^{-1}(z)P(z+1) - I = O(z^{-2})\), \( F(z) = O(z^{-2}) \) and

\[
\det(\lambda I + \hat{A}) = \det(\lambda I + \hat{B} + F) = \det(\lambda I + \hat{B}) + \sum (\lambda I + \hat{B}, F),
\]

where \( \sum(\lambda I + \hat{B}, F) \) represents the sum of all determinants formed from \( k \) rows of \( \lambda I + \hat{B} \) and \( n-k \) rows of \( F \) with natural ordering for \( 0 \leq k < n \).

If, for a particular determinant, \( m \) rows have been taken from \( F \), \( 1 \leq m \leq n \), there will possibly be a nonzero coefficient of \( \lambda^k \) for \( k \leq n-m \) which will be the sum of products of \( n-m-k \) elements from \( \hat{B} \) and \( m \) elements from \( F \). Since \( \hat{B} = O(z^q) \) and \( F = O(z^{q-2}) \), the coefficient of \( \lambda^k \) will have an order not exceeding

\[
(n - m - k)q + m(q - 2) = (n - k)q - 2m \leq (n - k)q - 2.
\]
Hence, $\sum (\lambda I + \hat{B}, F) = f_1 \lambda^{n-1} + f_2 \lambda^{n-2} + \cdots + f_n$, where

$$f_k = O(z^{q-2}), \quad k = 1, \cdots, n.$$  \hfill (4.2)

Combining (4.1) and (4.2) we have

$$\det(\lambda I + \hat{A}) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

where $a_k(z) = b_k(z) + f_k(z) = O(z^{q-2} + z^{-k})$ which gives the correct order estimates for $a_k$ when $q = 0$ and for $a_k, k = 1, \cdots, n-1$ when $q \geq 1$; but $a_n(z) = O(z^{nq-2}), \quad n \geq 2$. To obtain the order estimate $a_n(z) = O(z^{(n-1)q-2}), \quad q \geq 1$, we utilize a special property of systems with a regular singularity; namely, if $w(z+1) = A(z)w(z)$ has a regular singularity, there exists a fundamental matrix of the form $W(z) = S(z) z^R$ and hence

$$\det A(z) = \det S(z+1) \det(1 + z^{-1}) R[\det S(z)]^{-1} = 1 + O(z^{-1}).$$

Consider

$$\det(\lambda I + A) = \det[(\lambda + 1) I + \hat{A}]$$

$$= (\lambda + 1)^n + a_1 (\lambda + 1)^{n-1} + \cdots + a_n.$$  

Hence $a_n = \det A - 1 - a_1 - \cdots - a_{n-1}$ and using the preceding order estimates, we obtain $a_n(z) = O(z^{(n-1)q-2}), \quad n \geq 2$, which concludes the proof of Theorem 1.

5. Proof of Theorem 2. As in the proof of Theorem 1, we have ($q \geq 0$)

$$B(z) \sim A(z) = P(z+1)B(z)P^{-1}(z)$$  \hfill (5.1)

and

$$C(z) \sim B(z) = z^{-D}(1 + z^{-1})^{-D}C(z)^D$$  \hfill (5.2)

where $C(z) = I + O(z^{-1})$.

Since $P_0$ is nonsingular, from (5.1) we have

$$A_0 = P_0 \hat{B}_0 P_0^{-1}, \quad A_1 = P_0 \hat{B}_1 P_0^{-1} - \hat{A}_0 P_1 P_0^{-1} + P_1 P_0^{-1} \hat{A}_0.$$  \hfill (5.3)

From (5.2) we see that $\hat{B}(z) = z^{-D}G(z)z^D$, where $G(z) = O(z^{-1})$. Thus the $ij$th element of $\hat{B}(z)$ satisfies

$$\hat{b}_{ij}(z) = O(z^{d_i-d_i-1}).$$

Since the $d_i$ are nondecreasing, all the elements on and below the diagonal are zero for $\hat{B}_0$ if $q = 0$, and for $\hat{B}_0$ and $\hat{B}_1$ if $q \geq 1$. Thus $\hat{B}_0$ and hence also $\hat{A}_0$ is nilpotent and trace $(\hat{B}_0^k \hat{B}_0) = 0, \quad k = 0, \cdots, n-1$
for \( q \geq 1 \). For \( q = 0 \), write \( \hat{B}_0 = (b_{0i}^0) \), \( \hat{B}_1 = (b_{0i}^1) \) and note that \( b_{ij}^1 \neq 0, i > j \), implies \( d_k = d_k^0 \) for \( j \leq k < i \) and \( b_{ki}^0 = O(\varepsilon^{-1}) \) and hence \( b_{ij}^0 = 0, j \leq k < i \).

Since \( b_{ij}^0 = 0, i \geq j \), we have for \( k = 1, \ldots, n-1 \)

\[
\text{trace}(\hat{B}_0^{k} \hat{B}_1) = \sum_{i=1}^{n} \sum_{i < i_1 < \cdots < i_k} b_{i_1}^0 \cdots b_{i_{k-1}}^0 i b_{i_k}^1 = 0.
\]

Using equation (5.3) we have

\[
A_0^k A_1 = P_0 \hat{B}_0^k P_0^{-1} - A_0^{k+1} P_0^{-1} + A_0^k P_1 P_0^{-1} A_0.
\]

Thus, \( \text{trace} (\hat{A}_0^k \hat{A}_1) = \text{trace} (\hat{B}_0^k \hat{B}_1) = 0 \), and Theorem 2 is proved.

6. Proof of Theorem 3. \( \hat{A} \) satisfies its characteristic equation. Hence using the order conditions on the symmetric functions we obtain \( \varepsilon^{-nq} \hat{A}^n = O(\varepsilon^{-2}), q \geq 1 \) and \( \hat{A}^n - (\text{trace } \hat{A}) \hat{A}^{n-1} = O(\varepsilon^{-2}), q = 0 \), or \( \hat{A}_0^0 = 0 \) and

\[
A_0^{n-1} A_1 + A_0^{n-2} A_1 A_0 + \cdots + A_1 A_0^{n-1} = 0, \quad q \geq 1,
\]

(6.1)

\[
= (\text{trace } A_1) A_0^{n-1}, \quad q = 0.
\]

Since \( \hat{A}_0^n = 0 \) and \( \hat{A}_0^{n-1} \neq 0 \) by hypothesis, there exists a nonsingular matrix \( G \) such that \( \hat{A}_0 \) has Jordan canonical form \( N = G^{-1} \hat{A}_0 G \) with 1 on the superdiagonal and 0 elsewhere. Setting \( G_1 = G^{-1} \hat{A}_1 G \), equation (6.1) becomes

\[
N^{n-1} G_1 + N^{n-2} G_1 N + \cdots + G_1 N^{n-1} = 0, \quad q \geq 1,
\]

\[
= (\text{trace } A_1) N^{n-1}, \quad q = 0.
\]

A simple computation using the special form of \( N \) shows that trace \( (N^k G_1) = 0, k = 0, 1, \ldots, n-1 \) for \( q \geq 1 \) and \( k = 1, \ldots, n-1 \) for \( q = 0 \); but trace \( (N^k G_1) = \text{trace} (\hat{A}_0^k \hat{A}_1) \) and Theorem 3 is proved.

Remark. The equation (6.1) is always satisfied if the order conditions \( a_k = O(\varepsilon^{q-2} + \varepsilon^{-k}) \) are satisfied. However, if \( \hat{A}_0 \) is not nilpotent of maximum rank, this equation does not imply that \( \text{trace} (\hat{A}_0^k \hat{A}_1) = 0 \), \( k = 1, \ldots, n-2 \).

7. Proof of Corollary. The necessity can be proved as in Theorem 1 and is omitted. To prove sufficiency consider the equation

\[
(I + P_1(z + 1)^{-1})^{-1} A(z)(I + P_1-z^{-1})
\]

\[
= z^q A_0 + z^{q-1} \{ A_0 P_1 - P_1 A_0 + A_1 \} + O(z^{q-2}).
\]

Thus, the sufficiency reduces to showing that the equation \( \hat{A}_0 P_1 - P_1 \hat{A}_0 + \hat{A}_1 = 0 \) has a solution. It is well known that if \( \hat{A}_0 \) is
nilpotent of maximum rank, i.e. \( \hat{A}_0^n = 0, \hat{A}_0^{n-1} \neq 0 \), then trace \((\hat{A}_0^k \hat{A}_1) = 0, k = 0, 1, \ldots, n-1\) is necessary and sufficient for a solution of this equation (for a proof of this fact with this formulation see Wasow [7, pp. 102–104]). Since these conditions are satisfied by Theorem 3, the Corollary is proved.

8. Example. Let \( N \) be a maximum rank nilpotent in Jordan form as given in §6 and \( R \) a constant diagonal matrix. Then the system \( u(z + 1) = B(z)u(z) \) where \( B = z^a N + I + z^{-1}R \) has a regular singularity. This is easily seen since \( (z + 1)D B(z) z^{-D} = I + O(z^{-1}) \) where \( D = \text{diag}(0, q + 1, 2(q + 1), \ldots, (n-1)(q+1)) \).

For any constant matrix \( E \), let \( P(z) \) be a solution to the equation

\[
P(z + 1) = (I + Ez^{-2})^{-1}P(z), \quad P(z) = I + O(z^{-1})
\]

(this is a special case of a singularity of the first kind, see Harris [2]). The system \( w(z + 1) = A(z)w(z) \) has a regular singularity if \( A(z) \) is defined as

\[
(8.1) \quad A(z) = P^{-1}(z + 1)B(z)P(z).
\]

It follows that \( \hat{A}(z) = P^{-1}(z) [\hat{B}(z) + z^{-2}EB(z)]P(z) \) and hence

\[
\det(\lambda I + A) = \det(\lambda I + \hat{B} + z^{-2}EB(z)).
\]

Choose the first \( n - 1 \) rows of \( E \) to be zero and the \( n \)th row to be \( (1, 1, \cdots, 1, 0) \), \( R = \text{diag} (0, \cdots, 0, 1) \) and note that \( \hat{B} + z^{-2}EB = z^a N + z^{-2}EN + z^{-2}E + z^{-1}R \).

If \( D_n(\lambda) = \det(\lambda I + \hat{A}) \) where \( \hat{A} \) is \( n \) by \( n \), then considering \( n \) as a variable, \( n \geq 2 \), it follows that

\[
D_{n+1}(\lambda) = \lambda D_n(\lambda) + (-1)^{n+1}z^{a-2}\lambda + (-1)^{n+2}z^{a-2}
\]

and hence by induction that

\[
D_n(\lambda) = \lambda^n + (z^{a-2} + z^{-1})\lambda^{n-1} + \sum_{k=2}^{n-1} (-1)^{k+1}(z^{kq-2} + z^{(k-1)q-2})\lambda^{n-k} + (-1)^{n+1}z^{(n-1)q-2},
\]

and hence the order conditions are sharp for all \( k, n \) and \( q \geq 0 \).

References

2. W. A. Harris, Jr., Linear systems of difference equations, Contributions to Differential Equations 1 (1963), 489–518.


University of Minnesota