MORE CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Let $X$ be a real inner product space of dimension at least three and let $M$ be a 2-dimensional subspace of $X$. For a vector $u$ in $X$ but not in $M$, let $v$ be the vector in $M$ closest to $u$. It is easily seen that (i) if $v = 0$, then all of the vectors of norm 1 in $M$ are equidistant from $u$ and (ii) if $v \neq 0$ and $w = |v|^{-1}v$, then of all the vectors of norm 1 in $M$ $w$ is the closest to $u$. The purpose of this paper is to show that each of these properties characterize those normed linear spaces which are inner product spaces. (For a survey of such results, see [3, pp. 115-121].)

Throughout, we let $E$ denote real Euclidean 3-space. Our proofs are based on the following two characterizations of ellipsoids in $E$. Theorem A is due to G. Birkhoff [1]. Theorem B is due to Marchaud [4] and generalizes a result due to Blaschke [2, pp. 157-159].

(A) Let $K$ be a compact convex body in $E$ with bounding surface $S$. Suppose there exists a point 0 interior to $K$ satisfying: for any line $m$ through 0 and point $P$ in $m \cap S$, if $M$ is a plane through 0 so that its translate through $P$ supports $K$, then for skew cylindrical coordinates $(r, \theta, z)$ with $m$ the line $r = 0$ and $M$ the plane $z = 0$, the equation of $S$ is of the form $r = f(z) \cdot g(\theta)$. Then $K$ is an ellipsoid.

(B) Let $K$ be a compact convex body in $E$ with bounding surface $S$ satisfying: for every direction $d$ in $E$, there exists a corresponding plane $M_d$ such that the cylinder in the direction $d$ generated by the plane curve $S \cap M_d$ circumscribes $K$. Then $K$ is an ellipsoid.

Theorem. Let $X$ be a real normed linear space of dimension at least three. If $X$ satisfies either condition (1) or (2) below, then $X$ is an inner product space.

(1) For every 2-dimensional subspace $M$ of $X$ and vector $u$ not in $M$ for which $|u| = \min \{ |u-w| : w \in M \}$, we have $|u-w| = |u-w'|$ for all $w$ and $w'$ in $M$ with $|w| = |w'| = 1$.

(2) For every 2-dimensional subspace $M$ of $X$ and vector $u$ not in $M$ for which there exists a vector $v$ in $M$, $v \neq 0$, satisfying $|u-v| = \min \{ |u-w| : w \in M \}$, we have $|u-v|^{-1}v = \min \{ |u-w| : w \in M, |w| = 1 \}$.

Received by the editors October 6, 1966 and, in revised form, June 27, 1967.

\footnote{Research supported in part by the National Science Foundation under grant GP 5707.}
Proof. It suffices to show that for any 3-dimensional subspace \( Y \) of \( X \) and any one-to-one linear mapping of \( Y \) onto \( E \), the image \( K \) of the unit ball in \( Y \) is an ellipsoid (cf. [3]). For simplicity, we shall assume that \( Y = E \). The first of the above conclusions follows from Theorem A and the second from Theorem B. The arguments are similar and we furnish only the latter.

Given a direction \( d \), let \( m \) be the line through 0 (the origin) in the direction \( d \). Let \( N \) be any plane containing \( m \), let \( n \) be a line in \( N \) parallel to \( m \) which supports \( K \cap N \), and let \( x \) be any point of \( K \cap n \). Let \( N' \) be a plane parallel to \( N \) which supports \( K \) and let \( y \) be any point of \( K \cap N' \). By the symmetry of \( K \), the plane \( N'' \) parallel to \( N \) and containing \( -y \) will also support \( K \). We wish to show that the 2-dimensional subspace \( M \) of \( E \) spanned by \( x \) and \( y \) has the desired property of \( M_d \) in (B). Thus, for \( S = \text{boundary} \ K \), we need to show that for any point \( z \) in \( S \cap M \), the line \( p \) through \( z \) parallel to \( m \) supports \( K \). Suppose \( z = ax + by \) and \( |z| = 1 \). If \( a = 0 \) or \( b = 0 \), then \( z = \pm x \) or \( \pm y \) and it is clear that \( p \) has the desired property. Assume \( a, b \neq 0 \); by the symmetry of \( K \), we need only consider \( a > 0 \). Let \( u = (1/2)x - (b/2a)y \) and let

\[
K_z = \{ w : w \text{ in } E, \ |w - u| \geq (1/2a) \}.
\]

If \( T \) is the mapping in \( E \) defined by \( T(w) = (1/2a)w + u \), then \( T(0) = u \), \( T(K) = K_z \) and \( T(z) = x \). Since \( T \) is a magnification followed by a translation to show that \( p \) supports \( K \) at \( z \), it suffices to show that \( n \) supports \( K_z \) at \( x \).

We note that the ball centered at \( u \) of radius \( |b|/2a \) is supported by \( N \) at \( v = (1/2)x \). Thus, \( v \) is in \( K_z \) and by condition (2), \( |u - x| = \min \{ |u - w| : w \text{ in } N, |w| = 1 \} \). Suppose \( n \) does not support \( K_z \). Then there must be a point \( w_0 \) common to \( n \) and \( i(K_z) \), the interior of \( K_z \). It follows that all points of the segment \( [w_0, v] \), except possibly \( v \), belong to \( i(K_z) \). But the segment \( [w_0, v] \) must contain a vector \( w_1 \) of norm 1. We then have \( |u - w_1| < |u - x| \), a contradiction. Thus, \( n \) supports \( K_z \) at \( x \) and our conclusion follows.

References


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