CURVATURE OF NONLINEAR CONNECTIONS

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1. Introduction. The curvature of a homogeneous (nonlinear) connection on a vector bundle (and in particular, on the tangent bundle) has been defined by several authors in terms of local reference frames. (See [1] and the references listed there.)

In this note, an intrinsic definition is given for the curvature of a general nonlinear connection on a smooth ($C^\infty$) vector bundle (modeled on a Banach space [3]).

Let $p: E \to X$ be a smooth vector bundle over a smooth manifold $X$. A (general nonlinear) smooth connection on it [5] is a smooth splitting of the (direct) exact sequence

$$0 \to VE \to TE \to p^{-1}TX \to 0$$

of vector bundles over the manifold $E$ (where $p^{-1}TY$ denotes the pullback, $p'$ is defined by factoring the tangent map $p*: TE \to TX$ through $p^{-1}TX$, and $VE$ is the kernel of $p'$ (or of $p_*$) with $J$ its inclusion).

The splitting is given by a smooth morphism $V: TE \to VE$ such that $VJ = I$, or equivalently by a smooth morphism $W: p^{-1}TX \to TE$ such that $p'W = I$. $V$ and $W$ (the left and right splitting maps, respectively) are related by $JV + Wp' = I$. In other words, $V = JV$ and $H = Wp'$ are the projection maps of a direct sum decomposition $TE = HE \oplus VE$ (where $HE = \text{kernel } W = \text{image } V$). Of course $HE$, the horizontal bundle, is isomorphic to $p^{-1}TX$.

On the other hand, the vertical bundle, $VE$, is canonically isomorphic to $p^{-1}E$. Hence there is a canonical map $r: VE \to E$ over $p$ (isomorphic on the fibres). $D = rV: TE \to E$ is the connection map. It is a smooth morphism of the tangent bundle structure on $TE$, and is also fibre-preserving for the fibres of the other fibre bundle structure, $p_*: TE \to TX$, on $TE$. If $D$ is linear on the $p_*$ fibres, one has a linear connection; if $D$ is merely 1-homogeneous, then a homogeneous connection (also called a nonlinear connection by most authors).

Remark. If $E_0 \subset E$ is an open set, a smooth splitting of (1) restricted over $E_0$ is a smooth connection on $E_0$. This added generality is needed for strictly nonlinear connections. Namely, a smooth homogeneous connection is always assumed to be on $E_0 = E - 0$ (because one on $E$ is necessarily linear).

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The left splitting map $V$ is a vertical bundle valued 1-form on the manifold $E$; hence $V$ is the connection form. The curvature form is its exterior derivative, $dV$, taken with respect to the linear Berwald connection on the vertical bundle, which will be shown to exist next.

2. The Berwald connection. Let $U$ be the domain of a chart on $X$ (and identify it with its homeomorphic image in the model space $B$ of $X$); let $TX\mid U \approx U \times B$ be the tangent chart. Suppose $E\mid U \approx U \times E$ is a chart on $E$; by taking tangent maps one gets a chart $TE\mid (E\mid U) \approx U \times E \times B \times E$. The sequence (1) over $E\mid U$ becomes the sequence

$$0 \to U \times E \times 0 \times E \to U \times E \times B \times E \to U \times E \times B \to 0$$

of bundles over $U \times E$, with $p'(x, a, \lambda, b) = (x, a, \lambda)$. The map $r: VE \to E$ is locally $r(x, a, 0, b) = (x, b)$ and $V(x, a, \lambda, b) = (x, a, 0, b + \omega(x, a)\lambda)$, whence $D(x, a, \lambda, b) = (x, b + \omega(x, a)\lambda)$.

Here the smooth map $\omega: U \times E \to L(B, E)$ is the local component of the connection for this chart [5]. The connection is linear or homogeneous iff each $\omega$ is, in its second variable, $a$. In the linear case $\Gamma(x)\langle a, \lambda \rangle = \omega(x, a)\lambda$ defines a smooth map $\Gamma: U \to L^2(E, B; E)$, the local Christoffel component.

Note if the connection is on $E_0 \subset E$, $\omega$ is defined on $U \times E_0 \subset U \times E$; e.g. $E_0 = E - 0$ in the homogeneous case. Let $\partial_i$ denote the $i$th partial (Fréchet) derivative (written as $D_i$ in [3, Chapter 1]).

**Proposition.** For a smooth connection on $E_0 \subset E$, the maps $\Omega: (U \times E_0) \times E \to L(B \times E, E)$ defined by $\Omega((x, a), b)(\mu, c) = \partial_\mu \omega(x, a)(b)c$ are the local components of a linear connection on $VE\mid E_0 \to E_0$ (the Berwald connection).

**Proof.** If $P(x, a) = (fx, A(x)a)$ is a change of charts $U \times E \approx U \times E$ on $E$, the corresponding change of charts $P_*$ on $TE$ is given by

$$P_*(x, a, \lambda, b) = (fx, A(x)a, f'(x)\lambda, A'(x)(\lambda)a + A(x)b)$$

with primes denoting (Fréchet) derivatives. The local equation for $D$ shows that the old and new local components $\omega$ and $\tilde{\omega}$ are related by the classical equation

$$A'(x)(\lambda)a + \tilde{\omega}(fx, A(x)a)f'(x)\lambda = A(x)\omega(x, a)\lambda.$$

Differentiating partially with respect to $a$ in direction $b$ and setting $\lambda = \mu$ produces

$$A'(x)(\mu)b + \partial_2\tilde{\omega}(fx, A(x)a)(A(x)b)f'(x)\mu = A(x)\partial_\mu \omega(x, a)(b)\mu.$$
Since locally $T(VE) \approx (U \times E \times 0 \times E) \times B \times E \times 0 \times E$, a connection map for a connection on $VE | E_0 \rightarrow E_0$ is locally $D(x, a, 0, b; \mu, c, 0, d) = (x, a; 0, d + \Omega((x, a), b)(\mu, c))$, where $\Omega: (U \times E) \times E \rightarrow L(B \times E, E)$ is smooth. The induced change of charts $P_{**}$ on $T(VE)$ is $P_{**}(x, a, 0, b; \mu, c, 0, d) = (fx, A(x)a, 0, A(x)b; f'(x)\mu, A'(x)(\mu)a + A(x)c, 0, A'(x)(\mu)b + A(x)d)$. Hence the old and new $\Omega$ and $\bar{\Omega}$ are related by

$$A'(x)(\mu)b + \bar{\Omega}((fx, A(x)a), A(x)b)(f'(x)\mu, A'(x)(\mu)a + A(x)c) = A(x)\Omega((x, a), b)(\mu, c).$$

For $\Omega$ defined by $\Omega((x, a), b)(\mu, c) = \partial_2\omega(x, a)(b)\mu$, (2) says this equation is satisfied. Note $\Omega$ is linear in its second variable, $b$. q.e.d.

Remark 1. A conceptual existence proof goes as follows. Let $V_1: T(TE) \rightarrow V(TE)$ be the connection form of the induced connection on $p_*: TE \rightarrow TX$, [5, Theorem 1], where $V(TE) = \text{kernel } p_*$. Since $VE \subset TE$, $V_1$ restricts to a map $V_1: T(VE) \rightarrow V(TE) | VE$. Now this latter bundle is canonically isomorphic to the bundle $v_1^{-1}(v^{-1}VE)$, where $v: VE \rightarrow E$ and $v_1: v^{-1}VE \rightarrow VE$. But $v^{-1}VE \approx V(VE) = \text{kernel } v$. Hence there is a canonical epimorphism $g: V(TE) | VE \rightarrow V(VE)$, and $gV_1$ is the connection form of a linear connection on $v: VE \rightarrow E$.

Remark 2. The original Berwald connection occurs in the case $E = TX$ and the connection is the canonical (homogeneous) connection of a smooth Finsler structure on $X$. (See the references in [1].)

3. The curvature form. For a smooth connection on the vector bundle $F \rightarrow X$, the covariant derivative $D_uA$ for $A$ a smooth section of $F$ and $u$ a vector field on $X$ is defined to be $DA_*u$ [2]. If the connection is linear, the exterior derivative of a smooth $F$-valued $r$-form $M$ on $X$ (i.e. a smooth antisymmetric section of $L^r(TX; F) \rightarrow X$) is the smooth $F$-valued $(r+1)$-form $dM$ on $X$ defined by

$$dM(u_0, u_1, \ldots, u_r) = \sum_{i=0}^{r} (-1)^i D_{u_i}M(u_0, \ldots, \hat{u}_i, \ldots, u_r)$$

$$+ \sum_{i<j} (-1)^{i+j}M([u_i, u_j], \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_r),$$

where $u_i$ are smooth vector fields on $X$. (Smoothness of the Christoffel component $\Gamma(x)(_, _) = \omega(x, _)_-$ implies $dM$ is a smooth section.)

For the case $r = 1$ the above equation reduces to

$$dM(u, v) = D_uMv - D_vMu - M[u, v].$$

In terms of a local chart, $M$ is represented by a smooth map
and $dM$ is represented by the smooth map $U \rightarrow L^2(B; F)$ defined by

\begin{equation}
\text{dm}(x)(\lambda, \mu) = m'(x)(\lambda)\mu - m'(x)(\mu)\lambda + \omega(x, m(x)\mu)\lambda - \omega(x, m(x)\lambda)\mu.
\end{equation}

Now the connection form $V$ of a smooth connection on $E_0 \subset E$ is locally the smooth map $U \times E_v \rightarrow L^2(B \times E, E)$ defined by $V(x, a)(\lambda, b) = b + \omega(x, a)\lambda$. Using the Berwald connection on $VE|_{E_0 \rightarrow E_0}$, (4) implies that the curvature form $dV$ is locally

\begin{equation}
dV(x, a)((\lambda, b), (\mu, c)) = \omega'(x, a)(\lambda, -\omega(x, a)\lambda)\mu - \omega'(x, a)(\mu, -\omega(x, a)\mu)\lambda,
\end{equation}

where $(x, a) \in U \times E_v$. Clearly if either of $(\lambda, b)$ or $(\mu, c)$ is vertical, i.e. if $\lambda = 0$ or $\mu = 0$, then the right side is 0. Hence

**Lemma.** The curvature form $dV$ is horizontal, i.e. $dV(A, B) = dV(HA, HB)$ for all $A, B \in T_eE, e \in E_0$.

**Theorem (Structure equation).** If $A$ and $B$ denote vector fields on $E_0$, then $dV(A, B) = - V[H_A, H_B]$.

**Proof.** Immediate from (3) and the lemma.

An immediate consequence of the structure equation is the following generalization of a result of Sasaki for Riemannian connections [4, p. 343].

**Corollary.** $HE \subset TE$ (over $E_0$) is integrable iff $dV = 0$.

4. The curvature tensor field. The smooth isomorphisms $VE \cong p^{-1}E$ and $HE \cong p^{-1}TX$ are fibre-wise given by $r_e: (VE)_e \cong E_x$ and $p^*(e) : (HE)_e \cong T_xX$ (for $pe = x$). Their inverses are the vertical and horizontal lift maps [2], denoted $f^v_e = r^{-1}_e(f) \in (VE)_e$ and $u^H_e = p^*(e)^{-1}(u) = W_e(u) \in (HE)_e$, where $f \in E_x$ and $u \in T_xX$. These maps induce isomorphisms

\begin{equation}
L^2((HE)_e; (VE)_e) \cong L^2(T_xX; E_x) \text{ for } e \in E_0 \text{ with } pe = x.
\end{equation}

Since each $dV(e)$ is horizontal, it corresponds to an antisymmetric map $R_e(\cdot, \cdot) \in L^2(T_xX; E_x)$ by the above isomorphism. That is, it is defined as $R_e(u, v) = r(e)dV(u^H, v^H)$, or locally as

\begin{equation}
R(x)(\lambda, \mu) = \text{ the right side of (5), which by [3, Proposition 11, p. 8] is}
\end{equation}

\begin{align*}
&= \partial_1\omega(x, a)(\lambda)\mu - \partial_1\omega(x, a)(\mu)\lambda \\
&\quad + \partial_2\omega(x, a)(\omega(x, a)\mu)\lambda - \partial_2\omega(x, a)(\omega(x, a)\lambda)\mu.
\end{align*}

(This is equation (28) of [1, p. 138], with $\theta^h = dx^h$.)
The totality of maps $R_x(-,-): T_xX \times T_xX \times (E_x \cap E_0) \to E_x$ is the curvature tensor field. It does not in general define a tensor field on $X$ in the usual sense of $x \mapsto R_x$ being a section of a vector bundle over $X$, because of lack of linearity in the $e$-slot. But if the connection on $E_0$ is homogeneous, then the maps $R_x(u, v): E_x \to E_x$ are smooth and homogeneous. If the connection is linear, then these maps are continuous linear, and $x \mapsto R_x$ defines a smooth section of $L^e(TX, TX, E; E) \to X$, which is the usual tensor field on $X$ (because $\partial x\omega(x, a)(c)\lambda = \omega(x, c)\lambda$ and $\partial x\omega(x, a)(\lambda)\mu = \Gamma'(x)(\lambda)(a, \mu)$).

Remark. The structural equation and the definition of $D$ yield the following equation for the curvature tensor field of a general nonlinear connection:

$$R(u, v)e = -D[u^H, v^H]e \quad \text{for all } e \in E_0,$$

where $u, v$ are vector fields on $X$. (It is due to Dombrowski [2, p. 78 in the linear case.]

References


