In 1951, S. N. Mergeljan [1] proved that there exists a bounded holomorphic function \( f \) for which

\[
\int \int_{|z|<1} |f'(z)| \, dS = \infty.
\]

An obvious geometric interpretation of (1) is that the length \( l(r) \) of the image of the circle \( |z| = r \) grows so rapidly, as \( r \to 1 \), that \( l(r) \) is not an integrable function of \( r \).

An alternate geometric interpretation of (1) is that the length \( V(f, \theta) \) of the image of the radius of \( e^{i\theta} \) is not an integrable function of \( \theta \). W. Rudin [2, Theorem III] has proved a proposition stronger than Mergeljan’s, namely, that there exist Blaschke products \( B(z) \) such that \( V(B, \theta) = \infty \) for almost all \( \theta \). It follows that there exists a function \( f \), holomorphic in the unit disk \( D \) and continuous in the closure of \( D \), such that \( V(f, \theta) = \infty \) for almost all \( \theta \) [2, Theorem IV].

Both Mergeljan’s and Rudin’s arguments involve nonconstructive steps, and therefore they do not allow us to visualize the functions \( f \) in terms of any of the customary representations. In this note, I give two explicit constructions that prove Mergeljan’s result. Unfortunately, my examples are inadequate for Rudin’s theorem.

We begin with the function \( \frac{a^n - z^n}{1 - a^n z^n} \), where \( 2^{-1/n} < a < 1 \). We write \( a^n = \alpha \) and \( z^n = \xi \), and we observe that for \( 0 < \rho < \alpha \), the maximum and minimum values of \( |(\alpha - \xi)/(1 - \alpha \xi)| \) on the circle \( |\xi| = \rho \) are

\[
(\alpha + \rho)/(1 + \alpha \rho) \quad \text{and} \quad (\alpha - \rho)/(1 - \alpha \rho),
\]

respectively. The difference between the two moduli is \( 2\rho(1 - \alpha^2)/(1 - \alpha^2 \rho^2) \). Therefore the function \( \frac{a^n - z^n}{1 - a^n z^n} \), whose \( 2n \) points of maximum and minimum modulus on the circle \( |z| = r \) separate each other, maps that circle onto a curve of length greater than

\[
2n \cdot 2r^n \frac{(1 - a^{2n})/(1 - a^{2n}r^{2n})}{(1 - a^{2n})/(1 - a^{2n}r^{2n})} \quad (0 < r < a).
\]

The integral of this quantity, taken over the interval \( 3^{-1/n} < r < a \),

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is greater than $K_1 n (1 - a) \log n (1 - a)$, where $K_1$ is a constant independent of $a$ and $n$.

We now consider the Blaschke product

$$B(z) = \prod_{k} \frac{a_k - z}{1 - a_k z^{n_k}}.$$ 

The product converges if $\sum n_k (1 - a_k) < \infty$, in particular, if

$$n_k (1 - a_k) = 1/k (\log k)^{3/2} \quad (k = 2, 3, \ldots).$$

If moreover the sequence $\{n_k\}$ increases fast enough, we obtain disjoint intervals $r_k < r < a_k$ such that

$$\int_{r_k}^{a_k} \int_0^{2\pi} |B'(re^{i\theta})| r d\theta dr > K_2 n_k (1 - a_k) \log n_k (1 - a_k) > K_2 / k (\log k)^{1/2},$$

and Mergeljan's theorem is proved.

From our construction, we see immediately that there exists a continuous function $f$ satisfying condition (1). Indeed, it is sufficient to choose finite Blaschke products $B_m$ such that, for each of certain disjoint concentric annuli $A_m$,

$$\int \int_{A_m} |B_m'| dS - \sum_{j \neq m} \int \int_{A_j} |B_j'| dS > m^3,$$

and to take $f(z) = \sum m^{-2} B_m(z)$.

Our second example is based on the function

$$g(z) = \exp \left( -a \frac{1 + z^n}{1 - z^n} \right).$$

Since the maximum and minimum modulus of $g(z)$ on the circle $|z^n| = \rho$ are

$$\exp \left( -a \frac{1 - \rho}{1 + \rho} \right) \quad \text{and} \quad \exp \left( -a \frac{1 + \rho}{1 - \rho} \right),$$

the function $g$ maps the circle $C_r$ onto a curve of length greater than

$$2n \left\{ \exp \left( -a \frac{1 - r^n}{1 + r^n} \right) - \exp \left( -a \frac{1 + r^n}{1 - r^n} \right) \right\}.$$

To estimate the integral of this lower bound over the interval $0 < r < 1$,
we make the substitution $s = (1 - r^n)/(1 + r^n)$, and of the resulting integral

$$4 \int_0^1 (e^{-as} - e^{-a/s})(1 - s)^{-1+1/n}(1 + s)^{-1-1/n} ds$$

we discard everything except the portion over $(0, a^{1/2})$. We may then replace the algebraic factors by a constant, and the quantity to be determined is greater than

$$K_3 \int_0^{a^{1/2}} (e^{-a} - e^{-a/s}) ds.$$

Consider separately each of the intervals $[(j - 1)a, ja]$ $(j = 1, 2, \ldots, [a^{-1/2}])$. Since the minimum of the integrand in the $j$th interval is $e^{-a} - e^{-1/j} > - a + j^{-1} - j^{-2}$, the value of the integral is greater than

$$a \sum_{j=1}^{[a^{-1/2}]} [-a + j^{-1} - j^{-2}] > K_4 a \log a |.$$

Now, for $k = 2, 3, \ldots$, let $a_k = k^{-1}(\log k)^{-2/3}$, and let

$$f(z) = \exp \left(- \sum a_k \frac{1 + zn_k}{1 - zn_k} \right).$$

If $n_k \to \infty$ fast enough, then $f$ again has the desired properties.

**References**


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