ON THE ZEROS OF VAN VLECK POLYNOMIALS

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1. Introduction. It is known [4] that there exist $C_{n+p-2,p-2}$ polynomials $V(x)$ of degree $(p-2)$ such that corresponding to each such $V(x)$ the differential equation

$$\prod_{j=1}^{p} (x - a_j) \left[ y'' + \left( \sum_{j=1}^{p} \frac{\alpha_j}{x - a_j} \right) y' \right] + V(x)y = 0,$$

where all $\alpha_j > 0$ and $a_1 < a_2 < \cdots < a_p$, has a unique polynomial solution $S(x)$ of degree $n$. Such $S(x)$ and $V(x)$ are called Stieltjes and Van Vleck polynomials respectively [2]. It has been shown that the zeros of all such $S(x)$ and $V(x)$ lie in $(a_1, a_p)$ [1] and [5]. We have proved that each $V(x)$ can have at most two zeros in any interval $(a_r, a_{r+1})$, $2 \leq r \leq p-2$ and at most one zero in each of the intervals $(a_1, a_2)$ and $(a_{p-1}, a_p)$ [3]. We can, however, improve this result if we consider the distribution of the zeros of $V(x)$ in $k$ ($k > 1$) consecutive intervals. We intend to prove the following result:

**Theorem 1.** (a) Any $k$ ($k \leq p-1$) consecutive intervals $(a_j, a_{j+1})$, \ldots, $(a_{j+k-1}, a_{j+k})$ contain at most $k$ and at least $(k-1)$ zeros of $V(x)$, if $j = 1$ or $j = p-k$.

(b) Any $k$ ($k < p-1$) consecutive intervals $(a_j, a_{j+1})$, \ldots, $(a_{j+k-1}, a_{j+k})$ contain at most $(k+1)$ and at least $(k-1)$ zeros of $V(x)$, if $j \neq 1$ and $j \neq p-k$.

In §2 we prove a result analogous to Rolle's theorem for the zeros of the derivative of a function with real and simple zeros. In §3 we give the proof of Theorem 1.

2. Analogue of Rolle's theorem. As pointed out in an earlier paper [3b] we may assume that no $V(x)$ has a zero at any $a_j$, $1 \leq j \leq p$, for the modification in the proof in the other case will be the obvious one. We recall from [3b] the following

**Lemma 1.** The zeros of a Van Vleck polynomial $V(x)$ of degree $(p-2)$ and those of the corresponding $S(x)$ of degree $n$ are the zeros of the function

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\[ G(x) = \sum_{j=1}^{n-1} \frac{1}{x - x_j'} + \sum_{j=1}^{p} \frac{\alpha_j}{x - a_j}, \]

and conversely, where \( x_j' (1 \leq j \leq n-1) \) are the zeros of the derivative \( S'(x) \) of the \( S(x) \).

Let \( E \) denote the point set consisting of the zeros of \( S(x) \) and those of the corresponding \( V(x) \). Also, let \( F \) consist of the singular points \( a_1, a_2, \ldots, a_p \) and the zeros of the derivative \( S'(x) \) of the \( S(x) \). Then, we have the following analogue of Rolle's theorem for the sets \( E \) and \( F \).

**Theorem 2.** Between any two consecutive points of \( E \) lies one and only one point of \( F \).

**Proof.** We prove first that between two consecutive points of \( E \) lies one point of \( F \). We consider various cases separately.

**Case 1.** Let the two points of \( E \) under consideration be \( x_k \) and \( x_{k+1} \), both zeros of \( S(x) \). Since the zeros of \( S(x) \) are real and simple \([3a]\), it follows that one zero \( x_i' \) of \( S'(x) \) lies between \( x_k \) and \( x_{k+1} \). \( x_i' \) is a point of \( F \) and we have \( x_k < x_i' < x_{k+1} \).

**Case 2.** Let the two points of \( E \) be \( x_k \), a zero of \( S(x) \) and \( t_1 \), a zero of the corresponding \( V(x) \). Then, by Lemma 1, \( x_k \) and \( t_1 \) are both zeros of \( G(x) \). It is easy to see that \( G(x) \) has points of discontinuity at the points \( a_j \) \((1 \leq j \leq p)\) and at \( x_j' \) \((1 \leq j \leq n-1)\). Also \( G(x) \) is a continuously decreasing function of \( x \) in every interval of continuity. As \( G(x_k) = 0 \) and \( G(t_1) = 0 \), it follows that \( G(x) \) must have a discontinuity between \( x_k \) and \( t_1 \), for otherwise it would contradict the monotonic character of \( G(x) \). That point of discontinuity of \( G \), being either an \( a_j \) or a \( x_j' \), is a point of \( F \).

**Case 3.** Let the two points of \( E \) be \( t_k \) and \( t_{k+1} \), both zeros of \( V(x) \). It can be shown as in Case 2 above, that the interval \((t_k, t_{k+1})\) contains a point of discontinuity of \( G \) which is a point of \( F \).

We show that between any two consecutive points of \( F \) is a point of \( E \).

**Case a.** Suppose the two consecutive points of \( F \) are \( a_k, a_{k+1} \). Then \( G(x) \) decreases continuously from \(+\infty\) to \(-\infty\) as \( x \) moves from \( a_k \) to \( a_{k+1} \), because the interval \((a_k, a_{k+1})\) does not contain any \( x_j' \). Thus \( G \) vanishes precisely once in \((a_k, a_{k+1})\). Since \( E \) consists of the zeros of \( G \), the result follows in this case.

**Case b.** Let the two consecutive points of \( F \) be \( a_k, a_{k}' \), a zero of \( S'(x) \). For convenience, suppose that \( a_k < x_k' \). (In case \( a_k = x_k' \), then it can be shown that \( V(x) \) has also a zero at this \( a_k \) \([3b]\).) Then again,
$G(x)$ decreases continuously from $+\infty$ to $-\infty$ as $x$ varies from $a_k$ to $x'_i$. Thus the interval $(a_k, x'_i)$ contains a point of $E$, namely the zero of $G$ in $(a_k, x'_i)$.

**Case c.** Suppose the two consecutive points of $F$ are $x'_i$ and $x'_{i+1}$, both zeros of $S'(x)$. Then, since the zeros of $S(x)$ are all real and simple [3a], $S(x)$ has one zero in $(x'_i, x'_{i+1})$ and this completes the proof.

It may be noted that an $a_j$ and a zero $x'_i$ of $S'(x)$ may coincide. We have shown that this is possible if and only if this $a_j$ is also a zero of $V(x)$ [3b]. We show now that no two points of $E$ can coincide. It is known that zeros of $S(x)$ are simple and those of $V(x)$ are also distinct. It then suffices to prove the following

**Theorem 3.** No zero of a Van Vleck polynomial $V(x)$ is a zero of the corresponding Stieltjes polynomial $S(x)$.

**Proof.** If possible, let $x_k$ be a zero of both the polynomials $V(x)$ and $S(x)$. Let $S(x) = (x-x_k)T(x)$, then $T(x_k) \neq 0$ and $T(x)$ has all its zeros distinct. Also,

$$S'(x) = (x-x_k)T'(x) + T(x), \quad S''(x) = (x-x_k)T''(x) + 2T'(x),$$

$$S'''(x) = (x-x_k)T'''(x) + 3T''(x).$$

Thus,

$$S'(x_k) = T(x_k) \neq 0, \quad S''(x_k) = 2T'(x_k), \quad S'''(x_k) = 3T''(x_k).$$

Let $V(x) = (x-x_k)R(x)$. Then, equation (1.1) becomes,

$$S''(x) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{x-a_j}\right)S'(x) = -\frac{(x-x_k)^2T(x)R(x)}{\prod_{j=1}^{p} (x-a_j)}.$$  \hspace{1cm} (2.2)

Equation (2.2) shows that the function

$$F(x) = S''(x) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{x-a_j}\right)S'(x)$$

has a double zero at $x_k$. Thus, $F(x_k) = F'(x_k) = 0$. Let

$$T(x) = \prod_{j=1; j \neq k}^{n} (x-x_j),$$

then $x_j$ are real, distinct and $x_j \neq x_k$, $(j = 1, \ldots, n; j \neq k)$. Hence,

$$\frac{T'(x_k)}{T(x_k)} = \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j}$$  \hspace{1cm} (2.3)

and
\[
\frac{T''(x_k)}{T(x_k)} = \sum_{r<j=1; r \neq k}^{n} \frac{2}{(x_k - x_j)(x_k - x_r)}
\]
(2.4)

\[
= \left[ \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} \right]^2 - \sum_{j=1; j \neq k}^{n} \frac{1}{(x_k - x_j)^2}.
\]

Also,
\[
\frac{S''(x_k)}{S'(x_k)} = \frac{2T'(x_k)}{T(x_k)} = 2 \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j}.
\]

Now, \( F(x_k) = 0 \) gives
\[
2 \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} + \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j} = 0,
\]
or,
\[
2 \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} = - \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j}.
\]
(2.5)

Also, \( F'(x_k) = 0 \) gives,
\[
3T''(x_k) + 2 \left( \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j} \right) T'(x_k) - \left( \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} \right) T(x_k) = 0,
\]
or, since \( T(x_k) \neq 0, \)
\[
3 \frac{T''(x_k)}{T(x_k)} + 2 \left( \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j} \right) \frac{T'(x_k)}{T(x_k)} - \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0.
\]

Using equations (2.3) and (2.4), this reduces to
\[
3 \left[ \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} \right]^2 - 3 \sum_{j=1; j \neq k}^{n} \frac{1}{(x_k - x_j)^2}
\]
\[
+ 2 \left( \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j} \right) \left( \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} \right)
\]
\[
- \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0.
\]

In view of equation (2.5), this can be written as
\[
-3 \sum_{j=1; j \neq k}^{n} \frac{1}{(x_k - x_j)^2} - \left[ \sum_{j=1; j \neq k}^{n} \frac{1}{x_k - x_j} \right]^2
\]
\[
- \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0.
\]
(2.6)
In the left-hand side of equation (2.6), each term is negative and hence the sum cannot vanish. This contradiction leads to the desired result.

3. Proof of Theorem 1. Let the $k$ consecutive intervals $(a_j, a_{j+1}), \ldots, (a_{j+k-1}, a_{j+k})$ contain $r$ ($1 \leq r \leq n$) zeros of $S(x)$. Let these zeros be $x_{m+1} < \cdots < x_{m+r}$. Then the following situations arise:

**Case I.** The intervals $(a_j, x_{m+1})$ and $(x_{m+r}, a_{j+k})$ contain each one zero of $S'(x)$. In this case the interval $(a_j, a_{j+k})$ contains precisely $(r+1)$ zeros of $S'(x)$. This interval, therefore, contains $(k+r+2)$ points of the set $F$. By Theorem 2, therefore, there are exactly $(k+r+1)$ points of $E$ in these $k$ intervals. Since we have supposed that these intervals contain $r$ zeros of $S(x)$, it follows that the intervals $(a_j, a_{j+1}), \ldots, (a_{j+k-1}, a_{j+k})$ contain exactly $(k+1)$ zeros of $V(x)$.

**Case II.** The interval $(a_j, x_{m+1})$ contains one zero of $S'(x)$ and $(x_{m+r}, a_{j+k})$ does not contain any zero of $S'(x)$. In such a case, $S'(x)$ has $r$ zeros in $(a_j, a_{j+k})$ and hence $(a_j, a_{j+k})$ contains $(k+r+1)$ points of $F$. By Theorem 2, there are exactly $(k+r)$ points of $E$. Thus $(a_j, a_{j+k})$ contains $k$ zeros of $V(x)$.

**Case III.** Neither of the intervals $(a_j, x_{m+1})$ and $(x_{m+r}, a_{j+k})$ contains any zero of $S'(x)$. Here $(a_j, a_{j+k})$ contains $(r-1)$ zeros of $S'(x)$ and hence $(k+r)$ points of $F$. By Theorem 2, $(a_j, a_{j+k})$ contains exactly $(k+r-1)$ points of $E$ and hence $(k-1)$ zeros of $V(x)$.

In case $j=1$, the situation of Case I cannot prevail. For, in this case, $(a_1, x_m)$ does not contain any zero of $S'(x)$, because $x_m$ is the smallest zero of $S(x)$ in $(a_1, a_{k+1})$. Also, if $j=p-k$, the Case I is not possible, because $(x_m, a_p)$ does not contain any zero of $S'(x)$. In this case, $x_m$ is the largest zero of $S(x)$.

Finally we consider the case when the $k$ intervals $(a_j, a_{j+1}), \ldots, (a_{j+k-1}, a_{j+k})$ do not contain any zero of $S(x)$. In such a situation the interval $(a_j, a_{j+k})$ contains at most one zero of $S'(x)$ and hence at most $(k+2)$ points of $F$. By Theorem 2, therefore, $(a_j, a_{j+k})$ contains at most $(k+1)$ points of $E$, or equivalently, at most $(k+1)$ zeros of $V(x)$. It may be noted, however, that in this case $V(x)$ will have at least $k$ zeros in $(a_j, a_{j+k})$, and that being possible only when $S'(x)$ has no zero in $(a_j, a_{j+k})$. It is easy to see that when $j=1$ or $j=p-k$, $S'(x)$ has no zero in $(a_j, a_{j+k})$ and hence $V(x)$ has precisely $k$ zeros in this interval in such a case. The proof of Theorem 1 is now complete.

If we set in Theorem 1, $k=p-1$ and $j=1$ and note that each $V(x)$ is a polynomial of degree $(p-2)$ we have the following result due to Van Vleck [5].

**Theorem (Van Vleck).** All the zeros of $V(x)$ lie in $(a_1, a_p)$. 

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Bibliography


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