REALIZABILITY OF METRIC-DEPENDENT DIMENSIONS

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1. Introduction and results. In [2] and [3], K. Nagami and the author introduced functions $d_2$ and $d_3$ from the class of all $(X, \rho)$ into the nonnegative integers, where $X$ is a nonnull metrizable topological space and $\rho$ is a metric for $X$, consistent with the topology of $X$. Formal definitions are given in [3], and are condensed as follows.

DEFINITION. $d_2(X, \rho)$ is the smallest integer $n$ such that for every set of $n + 1$ pairs $C_1, C'_1; C_2, C'_2; \ldots; C_{n+1}, C'_{n+1}$ of closed subsets of $X$ with $\rho(C_i, C'_i) > 0$ for $i = 1, 2, \ldots, n + 1$ there exist closed sets $B_1, B_2, \ldots, B_{n+1}$ such that (i) $B_i$ separates $X$ between $C_i$ and $C'_i$ and (ii) $\cap_{i=1}^{n+1} B_i = \emptyset$.

DEFINITION. $d_3(X, \rho)$ is the smallest integer $n$ such that given any positive integer $m$ and $m$ pairs $C_1, C'_1; C_2, C'_2; \ldots; C_m, C'_m$ of closed subsets of $X$ with $\rho(C_i, C'_i) > 0$ for $i = 1, 2, \ldots, m$ then there exist closed sets $B_1, B_2, \ldots, B_m$ such that (i) $B_i$ separates $X$ between $C_i$ and $C'_i$ and (ii) no $n + 1$ of the sets $B_1, B_2, \ldots, B_m$ have a point in common (order $\{B_i: i = 1, 2, \ldots, m\} \leq n$).

We also consider metric dimension of $(X, \rho)$, denoted $\mu \dim(X, \rho)$, and defined as the smallest integer $n$ such that for all $\epsilon > 0$ the set of all $\epsilon$-balls has a refining cover of order $n + 1$ or less.

The following results are in [3]:
(i) for all $(X, \rho)$, $d_2(X, \rho) \leq d_3(X, \rho) \leq \mu \dim(X, \rho) \leq \dim X$,
(ii) for totally bounded $(X, \rho)$, $d_3(X, \rho) = \mu \dim(X, \rho)$, and
(iii) for all $n > 2$ there exists a space $(X_n, \rho)$ with $d_2(X_n, \rho) < d_3(X_n, \rho) \leq \mu \dim(X_n, \rho) < \dim X = n$.

No example is known having $d_3$ strictly less than $\mu$ dim.

The main result of the present paper is stated in Theorem 1. Using this theorem we can prove Theorem 2, which extends to the function $d_3$ (provided $X$ is separable) a result proved in [4] for the function $\mu$ dim.

**Theorem 1.** Let $(X, \rho)$ be a separable metric space. Then there exists a homeomorphism $h$ of $X$ into a subset of $I^\omega$ (the Hilbert cube) such that, letting $\sigma$ denote the $I^\omega$ metric
(i) $d_2(X, \rho) = d_2(h(X), \sigma)$ and
(ii) $d_3(X, \rho) = d_3(h(X), \sigma)$.

Thus for any separable $(X, \rho)$ there is a topologically equivalent totally bounded metric $\sigma$ which preserves $d_2$ and $d_3$.

Received by the editors July 31, 1967.

1 This research was supported in part by the National Science Foundation Grant GP-5919.
Theorem 2. For a separable metric space \((X, \rho)\), suppose \(d_3(X, \rho) = r\), \(\dim X = n\), and \(r < n\). Then for every integer \(k\) \((r \leq k \leq n)\) there exists a metric \(\rho_k\) for \(X\) such that

(i) \(\rho_k\) is topologically equivalent to \(\rho\), and
(ii) \(d_3(X, \rho_k) = k\).

Unsolved Problem. In the statement of Theorem 2, replace \(d_3\) by \(d_2\). Is the resulting statement true?

2. Proof of Theorem 1. Define \(r\) and \(s\) as follows: \(d_2(X, \rho) = r\), \(d_3(X, \rho) = s\). Then since \(d_2(X, \rho) \geq r\) there exist \(2r\) closed sets \(C_1, C'_1, C_2, C'_2, \ldots, C_r, C'_r\) such that (i) \(\rho(C_i, C'_i) > 0\) for \(1 \leq i \leq r\) and (ii) if for each \(i\) the closed set \(B_i\) separates \(X\) between \(C_i\) and \(C'_i\) then \(\cap_{i=1}^r B_i \neq \emptyset\). Similarly, since \(d_3(X, \rho) \geq s\), there is an integer \(m\) and \(2m\) closed sets \(C_{r+1}, C'_{r+1}, \ldots, C_{r+m}, C'_{r+m}\), such that (i) \(\rho(C_i, C'_i) > 0\) and (ii) if for each \(i\) \((r < i \leq r + m)\), the closed set \(B_i\) separates \(X\) between \(C_i\) and \(C'_i\) then \(\text{ord}\{B_i : r + 1 \leq i \leq r + m\} \geq s\).

For each positive integer \(i\) we define a function \(f_i: X \to [0, 1/i]\), and define \(h: X \to I^\omega\) by the formula

\[ h(x) = (f_1(x), f_2(x), \ldots) \in I^\omega. \]

Let \(\{\rho_i: i > r + m\}\) be a countable dense subset of \(X\) and make the following definitions:

\[ f_i(x) = \frac{\rho(x, C_i)}{i(\rho(x, C_i) + \rho(x, C'_i))} \quad (i \leq r + m); \]
\[ f_i(x) = \frac{\rho(x, \rho_i)}{i(1 + \rho(x, \rho_i))} \quad (i > r + m). \]

Letting \(\sigma\) be the usual metric in \(I^\omega\) we have, for \(x, y \in X\),

\[ \sigma(h(x), h(y)) = \left( \sum_{i=1}^\infty (f_i(x) - f_i(y))^2 \right)^{1/2}. \]

Note that for

\[ x \in C_i, \quad y \in C'_i, \quad f_i(y) - f_i(x) = f_i(y) = 1/i \]

so that

\[ \sigma(h(C_i), h(C'_i)) > 0. \]

2.1. Lemma. Let \(\delta > 0\) be the smaller of 1 and the minimum \(\rho(C_i, C'_i)/4\) for \(1 \leq i \leq r + m\). Let \(\epsilon\) be given such that \(0 < \epsilon < \delta\), and let \(\eta = \epsilon \delta /2\). Then if \(x, y \in X\) and \(\rho(x, y) < \eta\) it follows that \(|f_i(x) - f_i(y)| < \epsilon /2i\) for all \(i\), and \(\sigma(h(x), h(y)) < \epsilon\).
Proof. Fix \( x \) and \( y \) such that \( \rho(x, y) < \eta \). Fix \( i \leq r + m \). To simplify notation, introduce \( A, B, \alpha, \) and \( \beta \) by the definitions

\[
\rho(x, C_i) = A, \quad \rho(x, C'_i) = B, \quad \rho(y, C_i) = A + \alpha, \quad \rho(y, C'_i) = B + \beta.
\]

Then

\[
|\alpha| < \eta, \quad |\beta| < \eta, \quad A + B + \alpha + \beta > 4\delta - 2\eta > 2\delta,
\]

and (see (2))

\[
i \left| f_i(x) - f_i(y) \right| = \frac{A}{A + B} \left( \frac{A + \alpha}{A + B + \alpha + \beta} \right)
\]

\[
\leq \frac{A}{A + B} \frac{|\beta|}{2\delta} + \frac{B}{A + B} \frac{|\alpha|}{2\delta} < \frac{\epsilon \delta}{2\delta} = \frac{\epsilon}{2}.
\]

Thus for \( i \leq r + m \) we have \( \left| f_i(x) - f_i(y) \right| < \epsilon/2i \). If \( i > r + m \), from (3) it is trivial that \( i \left| f_i(x) - f_i(y) \right| \leq \rho(x, y) < \epsilon/2 \). Thus for all \( i \), \( \left| f_i(x) - f_i(y) \right| < \epsilon/2i \), and, using the fact that \( \sum_{i=1}^{\infty} 1/i^2 = \pi^2/6 < 4 \), we have \( \sigma(h(x), h(y)) < \epsilon \), from (4). This completes the proof of the lemma.

2.2. A sufficient condition that \( h : X \to h(X) \) be a homeomorphism is that for \( x \in X \), \( M \subseteq X \), (i) if \( \rho(x, M) = 0 \) then \( \sigma(h(x), h(M)) = 0 \), and (ii) if \( \rho(x, M) > 0 \) then \( \sigma(h(x), h(M)) > 0 \). Statement (i) follows trivially from Lemma 2.1. To prove (ii) suppose \( \rho(x, M) = d > 0 \), and fix \( i \) \((i > r + m)\) so that \( \rho(x, p_i) < d/4 \). Then for all \( y \in M \) we have \( \rho(y, p_i) > 3d/4 \), and

\[
f_i(y) - f_i(x) > \frac{3d/4}{i(1 + 3d/4)} - \frac{d/4}{i(1 + d/4)} = \epsilon > 0,
\]

with \( \epsilon \) independent of \( y \).

Thus

\[
\sigma(h(x), h(M)) \geq \inf \{ (f_i(y) - f_i(x)) : y \in M \} \geq \epsilon.
\]

2.3. Assertion. If \( C, C' \) are disjoint closed subsets of \( X \) and \( \rho(C, C') = 0 \), then \( \sigma(h(C), h(C')) = 0 \).

Proof. Let \( \delta, \epsilon, \) and \( \eta \) be given as in the hypothesis of Lemma 2.1. Fix \( x \in C, y \in C' \) so that \( \rho(x, y) < \eta \). Then \( \sigma(h(C), h(C')) \leq \sigma(h(x), h(y)) \leq \epsilon \), by Lemma 2.1. Thus \( \sigma(h(C), h(C')) = 0 \).

2.4. Conclusion of Proof of Theorem 1. In view of 2.3, it is evident from definitions that \( d_2(h(X), \sigma) \leq r \) and \( d_3(h(X), \sigma) \leq s \), because in \( (h(X), \sigma) \) there is no "new" pair \( C, C' \) at positive distance. On the other hand, the \( r + m \) pairs \( C_1, C'_1; \cdots; C_{r+m}, C'_{r+m} \), which
guarantee that $d_2(X, \rho) \geq r$, $d_3(X, \rho) \geq s$ remain at positive distance in $(h(X), \sigma)$ so $d_2(h(X), \sigma) \geq r$, $d_3(h(X), \sigma) \geq s$.

3. Proof of Theorem 2. We are given a separable metric space $(X, \rho)$ with $d_3(X, \rho) = r < n = \text{dim } X$. From Theorem 1 there is a topologically equivalent metric $\sigma$ for $X$ such that (i) $d_3(X, \sigma) = d_3(X, \rho)$ and (ii) $(X, \sigma)$ is totally bounded. Thus [3, Theorem 5] $d_3(X, \sigma) = \mu \text{ dim } (X, \sigma) = r$. Now in [4], in the proof of the main theorem, a finite number of continuous functions $f_1, f_2, \ldots, f_t$ are defined, $f_i: X \to [0, 1]$, and metrics $\sigma_1, \sigma_2, \ldots, \sigma_t$ for $X$ are defined by the formula

$$
\sigma_i(x, y) = \sigma_{i-1}(x, y) + |f_i(x) - f_i(y)|,
$$

where $\sigma_0 = \sigma$. These have the property that $\mu \text{ dim } (X, \sigma_i) \leq \mu \text{ dim } (X, \sigma_{i+1}) \leq \mu \text{ dim } (X, \sigma_i) + 1$, and $\mu \text{ dim } (X, \sigma_i) = n$. Thus for any $k$ ($r < k \leq n$) there exists $i(k)$ such that $\mu \text{ dim } (X, \sigma_{i(k)}) = k$. But in the present case, with $(X, \sigma_0)$ totally bounded, every $(X, \sigma_i)$ is totally bounded (see Hurewicz [1, p. 200]) so by [3] we have $d_3(X, \sigma_{i(k)}) = k$, and the proof is complete.

References


