HOMOTOPIC ARCS ARE ISOTOPIC

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The purpose of this note is to show that if \( \alpha \) and \( \beta \) are flat arcs in an \( n \)-manifold \( M \) \((n \geq 3)\) and \( \alpha \) and \( \beta \) are homotopic (with fixed endpoints), then there is an isotopy of \( M \) onto itself, leaving the endpoints of \( \alpha \) and \( \beta \) fixed, which carries \( \alpha \) onto \( \beta \). This result is actually an application of an unstated theorem in §6 of [2] and is of primary interest when \( n=3 \), as theorems of this kind are well known for larger values of \( n \).

This same sort of technique has been used by James Kister to show that each arc in an \( n \)-manifold is isotopic, with fixed endpoints, to a flat arc. Of course, in this case, the isotopy is not ambient.

An \( n \)-manifold is a topological space which may be covered by open sets, each of which is homeomorphic to \( E^n \), Euclidean \( n \)-dimensional space. An arc in an \( n \)-manifold \( M \) is flat if there exists a closed neighborhood \( A \) of \( \alpha \) and a homeomorphism \( h \) of \( A \) onto \( D^n \), the unit ball in \( E^n \), which carries \( \alpha \) onto a straight line interval. A homotopy of a space \( X \) in a space \( Y \) is a map \( f: X \times I \to Y \); we shall sometimes write \( f(t, x) \) for \( f(x, t) \) or use another closed real interval to replace \( I = [0, 1] \). We say that \( f \) is fixed on a subset \( S \) of \( X \) if \( f|_S = f|_S \) for each \( t, t' \in I \). If \( f_t \) is a homeomorphism for each \( t \), we call \( f \) an isotopy, and if each \( f_t \) is also surjective and \( f_0 \) is the identity, then \( f \) is an ambient isotopy of \( X \). A path in a space \( Z \) is a map of \( I \) into \( Z \). The paths \( \omega \) and \( \omega' \) are path homotopic in \( Z \) if there is a homotopy \( g \) of \( I \) in \( Z \), fixed on \( \{0, 1\} \) such that \( \omega = g_0 \) and \( \omega' = g_1 \). If two arcs are images of \( I \) by homeomorphisms which are path homotopic, we shall also call the arcs path homotopic. It may easily be seen that two arcs \( \alpha \) and \( \beta \) are path homotopic if and only if they have common endpoints and there exists a path \( \lambda \) such that \( \lambda \) maps \([0, \frac{1}{2}]\) homeomorphically onto \( \alpha \), \([\frac{1}{2}, 1]\) homeomorphically onto \( \beta \), and \( \lambda \) is path homotopic to a constant path.

Theorem. Suppose that \( n \geq 3 \), \( M \) is an \( n \)-manifold, and \( \alpha \) and \( \beta \) are path homotopic flat arcs in \( M \) with common endpoints \( p \) and \( q \). Then there exists an ambient isotopy \( h_t \) \((0 \leq t \leq 1)\) of \( M \), fixed on \( p \) and \( q \), such that \( h_1(\alpha) = \beta \). Furthermore, if \( K \) is a closed set such that \( \alpha \) and \( \beta \) are path homotopic in \( M - K \) then we may require that \( h_t \) be fixed on \( K \).

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Proof. Let $M$, $\alpha$, $\beta$, $p$, $q$, and $K$ be as in the hypothesis and let $\lambda$ be a path in $M$ such that $\lambda| [0, \frac{1}{2}]$ and $\lambda| [\frac{1}{2}, 1]$ are homeomorphisms onto $\alpha$ and $\beta$ respectively, $\lambda(0) = \lambda(1) = p$, and $\lambda$ is path homotopic in $M - K$ to the constant path at $p$.

Now let $A$ be a closed neighborhood of $\alpha$ in $M - K$ such that there exists a homeomorphism carrying $A$ onto $D^n$ and $\alpha$ onto a straight line interval, and let $B$ be a similar neighborhood of $\beta$ in $M - K$. Let $C$ be an $n$-cell in $M - K$ containing $q$ in its interior. Let $u$ and $v$ be numbers such that $0 < u < \frac{1}{2} < v < 1$ and $\lambda([u, v]) \subset \text{int } C$.

We next describe an ambient isotopy of $M$ which carries $\alpha$ onto $\beta$, leaves $K \cup \{q\}$ fixed, but which moves $p$. We will later modify this isotopy so as to leave $p$ fixed.

In $D^n$, any straight line interval in the interior may be shrunk through itself toward an endpoint by an ambient isotopy, fixed on that endpoint and on $\partial D^n$. Using this fact we may define an isotopy $g_t (0 \leq t \leq u)$ of $M$ onto itself such that:

$$g_0 = \text{identity}, \quad g_u(\alpha) \subset \text{int } C,$$

$$g_t | M - A = \text{identity}, \quad g_t(q) = q$$

and

$$g_t(p) = \lambda(t).$$

Now let $\alpha'$ be $g_u(\alpha)$ and $\beta'$ be $\lambda([\frac{1}{2}, v])$. Then $\alpha'$ and $\beta'$ are flat arcs interior to $C$ and sharing the endpoint $q$. Thus there is an ambient isotopy of $C$, fixed on $\partial C \cup \{q\}$, which takes $\alpha'$ to $\beta'$. This may be used to extend $g_t$ to the interval $0 \leq t \leq v$ so that it satisfies

$$g_v(\alpha) = \beta', \quad g_t | M - (A \cup C) = \text{identity},$$

and

$$g_t(q) = q.$$

Notice that $g_v(p) = \lambda(v)$.

Finally, by a reversal of the first type of isotopy, we extend $g_t$ to an isotopy of $M$ onto $M$ for $0 \leq t \leq 1$, such that

$$g_1(\alpha) = \beta, \quad g_t | M - (A \cup B \cup C) = \text{identity}, \quad g_t(q) = q$$

and

$$g_t(p) = \lambda(t) \quad \text{if } 0 \leq t \leq u \text{ or } v \leq t \leq 1.$$  

Now let $\omega(t)$ be the path $g_t(p)$. Notice that $\omega$ and $\lambda$ differ only for $u \leq t \leq v$, and for these values of $t$, $\omega(t)$ and $\lambda(t)$ both lie in the $n$-cell $C$. Therefore $\omega$ and $\lambda$ are path homotopic in $M - K$ and hence $\omega$ is path homotopic to $\lambda$.
homotopic in $M-K$ to the constant path at $p$. Now since $n \geq 3$ and $q \in \omega(I)$, $\omega$ is also path homotopic to the constant path at $p$ in $M-(K \cup \{q\})$. That is, there exists a homotopy $F: I \times I \rightarrow M-(K \cup \{q\})$ such that

$$F(s, 0) = \omega(s) = g_*(p)$$

and

$$F(s, 1) = F(0, t) = F(1, t) = p.$$

We now apply the technique in [2]. Let $\mathcal{C}$ be the space of all homeomorphisms of $M$ onto $M$ which leave $K \cup \{q\}$ pointwise fixed, where $\mathcal{C}$ has the compact open topology. Define $\pi: \mathcal{C} \rightarrow M-(K \cup \{q\})$ by $\pi(h) = h(p)$. It is well known that the triple $(\mathcal{C}, \pi, M-(K \cup \{q\}))$ is a fiber bundle.

Now an ambient isotopy of $M$ which is fixed on $K \cup \{q\}$ is simply a path in $\mathcal{C}$ beginning at the identity. In particular the path $\phi$ in $\mathcal{C}$ defined by $\phi(t) = g_t$ ($0 \leq t \leq 1$) satisfies $\pi \circ \phi = F_0$. Now by the homotopy lifting property, there is a homotopy $G: I \times I \rightarrow \mathcal{C}$ such that $\pi \circ G = F$ and $G_0 = \phi$. Since $\pi$ then maps each of $G(s, 1)$, $G(0, t)$, and $G(1, t)$ into $p$, these are all members of $\mathcal{C}$ which leave $p$ fixed. Now define $h_t$ ($0 \leq t \leq 1$) as follows:

$$h_t = G(0, 3t), \quad 0 \leq t \leq 1/3,$$
$$= G(3t - 1, 1), \quad 1/3 \leq t \leq 2/3,$$
$$= G(1, 3 - 3t), \quad 2/3 \leq t \leq 1.$$

Then $h$ is an ambient isotopy of $M$, $h_0 = g_0 =$ identity, $h_1(\alpha) = g_1(\alpha) = \beta$, and $h$ is fixed on $K \cup \{p, q\}$. This completes the proof.

We remark that if $M$ is simply connected, the conclusion of the theorem holds for any two flat arcs with common endpoints. Also, a slight modification of the argument yields a similar theorem if $M$ is a manifold with boundary, provided that one of the common endpoints of $\alpha$ and $\beta$ is interior to $M$. Finally, we remark that in [1] Feustel proves a theorem of this type for nonambient isotopies of arcs in 2-manifolds with boundary, where the endpoints of the arcs are on the boundary, and also gives an example to show that these theorems are false for 2-manifolds.

References


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