1. Introduction. In 1933 in a paper in the *Bulletin of the American Mathematical Society* J. M. Clarkson [1] discussed the following line involution: In a projective space of three dimensions over the field of complex numbers, let \( \pi_1 \) and \( \pi_2 \) be two planes intersecting in a line \( \mu \). In \( \pi_1 \) let \( H_1 \) be a harmonic homology whose axis, \( d_1 \), is distinct from \( \mu \) and whose center, \( C_1 \), does not lie on \( \mu \). In \( \pi_2 \) let \( H_2 \) be a harmonic homology whose axis, \( d_2 \), is distinct from \( \mu \) and whose center, \( C_2 \), does not lie on \( \mu \). The image of a general line, \( I \), meeting \( \pi_1 \) and \( \pi_2 \) in points \( L_1 \) and \( L_2 \), respectively, is the line \( I' \) determined by the points \( L_1' \) and \( L_2' \) which are the images of \( L_1 \) and \( L_2 \) under \( H_1 \) and \( H_2 \), respectively. Among other things, Clarkson showed that the order, \( m \), of this involution is two, that the order, \( i \), of its complex of invariant lines is zero, and that the image of a general plane field of lines is a bilinear congruence with distinct directrices. Also, although Clarkson did not note this fact, it follows readily from his work, or from the formula [2] \( k = m - 2i + 1 \) that the order, \( k \), of the complex of singular lines, or lines which meet their images, is three. Furthermore, in this involution the exceptional lines, or lines which do not have a unique image, are not \( E \)-singular, that is they do not meet the exceptional line in their respective image-families.

The particular involution investigated by Clarkson is not the only one with the properties that \( m = 2 \), \( i = 0 \), \( k = 3 \) and that the image of a general plane field of lines is a bilinear congruence with distinct directrices. In fact the line involution defined by any two harmonic homologies, \( H_1: (d_1, C_1) \) and \( H_2: (d_2, C_2) \), in distinct planes has these properties except in the following cases:

(i) \( d_1, d_2, \) and \( \mu \) coincide,

(ii) \( d_1 \) and \( d_2 \) intersect on \( \mu \), and \( C_1 \) and \( C_2 \) coincide on \( \mu \).

In (i), the involution reduces to the line involution defined by the skew homology whose axes are \( \mu \) and the line determined by \( C_1 \) and \( C_2 \). In (ii) the involution is the line involution defined by the central homology whose plane is the plane determined by \( d_1 \) and \( d_2 \) and whose center is the point on \( \mu \) in which \( C_1 \) and \( C_2 \) coincide. In what follows, we shall refer to any line involution defined by two harmonic homologies in distinct planes as Clarkson's involution except when the homologies are specialized as in (i) or (ii).
The purpose of this paper is to prove that Clarkson's involution is unique; more specifically we shall prove the following theorem:

**Theorem 1.** Clarkson's involution is the only line involution with the following properties:

(i) \( m = 2, i = 0, k = 3 \),

(ii) The image of a general plane field is a bilinear congruence with distinct directrices,

(iii) The exceptional lines are not \( E \)-singular.

In our proof we shall work primarily on the nonsingular hyperquadric, \( V^2_4 \), in \( S^5 \) onto which the lines of \( S^3 \) are mapped by the well-known device of interpreting the Plücker coordinates of the lines of \( S^3 \) as point coordinates in \( S^5 \). In thus studying line involutions in \( S^3 \) as point involutions of \( V^2_4 \) into itself, we shall for convenience refer to the planes of \( V^2_4 \) which are the images of plane fields of lines in \( S^3 \) as \( f \)-planes, and we shall refer to the planes of \( V^2_4 \) which are the images of bundles of lines in \( S^3 \) as \( b \)-planes.

2. **The mapping of a general \( f \)-plane of \( V^2_4 \).** Under the hypotheses of Theorem 1, the image of a general line of \( V^2_4 \) is a nonsingular conic, and the image of a general \( f \)-plane, \( \pi_f \), is a nonsingular quadric surface, \( Q \). Since two conics on \( Q \) intersect in two points, whereas two lines of \( \pi_f \) have a single point in common, it follows that the lines of \( \pi_f \) must be transformed into a family of conics on \( Q \) having a fixed point, \( P \), in common. Moreover, since \( P \) is a simple point on each of these conics, it must have a single pre-image on each line of \( \pi_f \). In other words, the total pre-image of \( P \) in \( \pi_f \) must be a line, \( p \).

The line \( p \), though not an ordinary line, must also have a conic image on \( Q \). Hence it must contain either one exceptional point with a conic image or two exceptional points each with a line of images. The former possibility can be rejected at once, because a line of \( \pi_f \) through such a point could not have a proper image curve, as it must since its points are general points of \( \pi_f \). Hence \( p \) contains two points, \( G_1 \) and \( G_2 \), each of which is transformed into a line on \( Q \). Moreover, since these lines each contain \( P \), they are, in fact, the generators, \( g_1 \) and \( g_2 \), of \( Q \) which pass through \( P \).

Since the image of \( G_i \) is the line \( g_i \), it follows that the image of a general line, \( l \), through \( G_i \) in \( \pi_f \) is a composite conic having \( g_i \) as one component. The other component, which is the proper image of \( l \), meets \( g_i \) and must therefore be a generator of the regulus, \( R_j \), to which \( g_i \) belongs. In other words, if \( R_1 \) and \( R_2 \) are the reguli of \( Q \) which contain \( g_1 \) and \( g_2 \), respectively, then the image of the pencil \( (\pi_f, G_i) \) is
the regulus $R_2$ and the image of the pencil $(\pi_f, G_2)$ is the regulus $R_1$. From this it is clear that the proper images of the conics of $\pi_f$ which contain $G_1$ and $G_2$ comprise the totality of conics on $Q$.

3. The $E$-point locus on $V_4^2$. Since the proper image of a general generator of the image quadric, $Q$, of a general $f$-plane, $\pi_f$, is a line in $\pi_f$, it follows that on each generator of $Q$ there must be one exceptional point, $E$, with a line of images. The locus of these $E$-points on $Q$ must therefore be either a proper conic or a pair of generators, one from each regulus. Moreover, since a general conic, $C$, through the point $P$ on $Q$ has a line in $\pi_f$ as its proper image, it follows that the two intersections of $C$ and the $E$-conic on $Q$ must be distinct from $P$. Hence the pre-image in $\pi_f$ of the $E$-conic on $Q$ cannot be a single line or the pair of points $G_1$ and $G_2$, but must be either a proper conic or a pair of lines, one on $G_1$ and one on $G_2$.

Since a general $f$-plane contains two and only two exceptional points, namely $G_1$ and $G_2$, it is clear that on $V_4^2$ the locus of $E$-points can be neither a complex nor a curve. Therefore the locus of $E$-points must be a surface, and the following possibilities exist:

(i) The $E$-locus contains $\infty^1$ $E$-conics each lying on the image quadrics of $\infty^2 f$-planes.

(ii) The $E$-locus contains $\infty^2$ $E$-conics each lying on the image quadrics of $\infty^1 f$-planes.

(iii) The $E$-locus contains $\infty^3$ $E$-conics each lying on the image quadric of a single $f$-plane.

The possibility (i) can clearly be rejected, for the pre-image of any particular $E$-conic, $C$, is a ruled surface necessarily lying in the $S_4$ which is tangent to $V_4^2$ at the point of intersections of any two of the $f$-planes whose image quadrics contain $C$. Hence every $f$-plane whose image contains $C$ must be in this $S_4$, which is impossible since a tangent $S_4$ contains only $\infty^1 f$-planes of $V_4^2$.

In (iii), if the $\infty^3 E$-conics are proper they necessarily lie on a non-composite quadric surface, while if they are composite their components necessarily lie, respectively, in two planes of $V_4^2$ with a line in common. Let $\pi_f$ and $\pi'_f$ be two $f$-planes intersecting in a general point $O$, and let $Q$ and $Q'$ be their respective image quadrics. In addition, let $C$ and $C'$ be the $E$-conics on $Q$ and $Q'$, respectively, and let $A$ and $B$ be the points common to $C$ and $C'$. Then if $O'$ is the image of $O$, the plane $\pi$ determined by $O'$, $A$, $B$ lies in both the $S_3$ containing $Q$ and the $S_3$ containing $Q'$. Thus $\pi$ has the same conic $C''$ in common with both $Q$ and $Q'$ and, perforce, every point of $C''$ has a pre-image in both $\pi_f$ and $\pi'_f$. Hence $C''$, which is surely distinct from the $E$-
conics $C$ and $C'$ since it contains the image of the general point $O$, is a second conic of $E$-points on both $Q$ and $Q'$, which is impossible. Thus (ii) remains the only possibility, that is, the $E$-locus contains $\infty^2$ $E$-conics each of which lies on the image quadrics of $\infty^1 f$-planes.

Now the $\infty^1 f$-planes which are the pre-images of the $\infty^1$ quadrics which have a particular $E$-conic in common can have no free intersection. Hence they must consist of all the $f$-planes through some point $F$; that is, they must comprise the totality of $f$-planes in the special linear complex, $\Gamma_F$, which is cut from $V^2_3$ by the tangent $S_4$ at $F$. Moreover $F$ must be the pre-image of an $E$-point, for otherwise the image quadrics of the planes of $\Gamma_F$ would all have an $E$-conic as well as the image of $F$ in common, and hence would coincide. If the $E$-conic under consideration is proper, its pre-image curve in each $f$-plane of $\Gamma_F$ is a proper conic containing $F$. Thus the pre-image locus of the $E$-conic, being a ruled surface in $\Gamma_F$ with conic directrices intersecting in a self-corresponding point, must be a cubic surface. Therefore the pre-image locus of the surface of $E$-points must be a $V^3_3$ since it is intersected by $S_4$'s in cubic surfaces. But if this is the case, then the $E$-point pre-image locus in a general $f$-plane must be a cubic curve, whereas it actually is only a conic. Thus the $E$-conic on the image quadric of a general $f$-plane cannot be proper but must, instead, be composite with a composite pre-image in that plane consisting of one line on each of the exceptional points $G_1$ and $G_2$ in that plane. Moreover, since each pre-image line of a component of any $E$-conic contains an $E$-point, it is clear that the $E$-locus is, in fact, contained in the $E$-point pre-image locus.

Now since the $E$-locus contains $\infty^2$ composite conics, it must be either a nonsingular quadric surface or a pair of planes. It cannot be the former, however, since each $f$-plane must contain two $E$-points. Hence the $E$-locus must consist of a pair of $f$-planes, $\pi_1$ and $\pi_2$, intersecting in a point $M$, and the $\infty^2$ composite $E$-conics are simply the conics consisting of an arbitrary line of the pencil ($\pi_1, M$) and an arbitrary line of the pencil ($\pi_2, M$). Moreover each of the linear complexes $\Gamma_1$ and $\Gamma_2$, comprising the pre-image of the $E$-surface, contains one of the plane components of the $E$-surface and hence is special.

Finally, since it is part of our hypothesis that the $E$-lines of our involution are not $E$-singular, it follows that $\pi_1$ does not lie in $\Gamma_1$ and $\pi_2$ does not lie in $\Gamma_2$. In other words, the $E$-point on the pre-image line of an $E$-point in $\pi_1$ lies in $\pi_j$.

4. The complex of singular elements. Since the image of a general $f$-plane is a nonsingular quadric surface $Q$, it is clear that a general
f-plane, \( \pi_f \), has either a single point or a line in common with its image. If the intersection is a line, its points cannot be invariant for this would imply the existence of a complex of invariant lines, whereas \( i = 0 \). On the other hand, the points of such a line cannot be exceptional, for a general f-plane contains only two E-points. Hence the line must be transformed into itself in such a way that precisely two of its points, \( I \) and \( I' \), are left invariant. Moreover, since the line \( II' \) is a generator of \( Q \) which is its own image, it follows that it must pass through one of the two exceptional points in \( \pi_f \), say \( G_1 \). The lines determined by the other exceptional point, \( G_2 \), and the invariant points \( I \) and \( I' \) are then transformed into generators, \( r_1 \) and \( r'_1 \), of the regulus \( R_1 \) on \( Q \). Now the component, \( e_2 \), of the E-conic on \( Q \) which belongs to \( R_2 \) meets \( r_1 \) and \( r'_1 \) in \( A \) and \( A' \), say, and has as its pre-image in \( \pi_f \) a line, \( e_2 \), on \( G_1 \). Furthermore, since E-points are not E-singular, it is clear that the E-point on \( e_2 \), namely \( G_1 \), does not lie in the E-plane which contains \( e_2 \), so that it is \( G_2 \) which lies in the same E-plane with \( e_2 \). Thus \( G_2A \) and \( G_2A' \) are lines of \( V_4^2 \), and hence the planes \( G_2IA \) and \( G_2IA' \) lie entirely on \( V_4^2 \). Therefore the lines \( G_2I \) and \( G_2I' \) are lines of singular points as, of course, is \( II' \).

The cubic complex of singular elements meets the quadric \( Q \) in a composite sextic consisting of \( r_1, r'_1, II' \) and a residual component of order three. Moreover, even if this is a composite cubic containing the generators whose pre-images in \( \pi_f \) are \( G_1 \) and \( G_2 \) there is at least a linear component whose pre-image in \( \pi_f \) is different from \( G_2I, G_2I', \) and \( II' \). But this is impossible since the intersection of \( \pi_f \) and the complex of singular elements is only a cubic curve. Hence \( \pi_f \) and its image quadric cannot have a line in common, and hence must intersect in a single point, which is necessarily invariant.

Let \( I \) be the invariant point common to a general f-plane \( \pi_f \) and its image quadric, \( Q \). Then the lines \( G_1I \) and \( G_2I \) are transformed respectively into the generators, \( r_2 \) and \( r_1 \), of \( Q \) which contain \( I \). Moreover, as before, the points of \( G_1I, G_2I, r_1, \) and \( r_2 \) are all singular, and the plane, \( \alpha_1 \), determined by \( G_1I \) and \( r_2 \) and the plane, \( \alpha_3 \), determined by \( G_2I \) and \( r_1 \) are b-planes on \( V_4^2 \).

Now in \( \alpha_1 \), since \( G_1I \) is transformed into \( r_2 \) in such a way that \( I \) is self-corresponding, it follows that the transformation is a perspectivity with center, \( S \), say. Let \( l \) be the line joining a general point, \( A \), on \( G_1I \) to its image, \( A' \), on \( r_2 \). Since the image of \( l \) must contain \( A \) and \( A' \), it is clear that if the image of \( l \) is a line, then every point of \( l \), and hence every point of \( \alpha_1 \), is singular with image in \( \alpha_1 \). Similarly, if the image of \( l \) is a conic lying in a b-plane, \( \pi_b \), or is an entire b-plane, \( \pi_b \), then \( \pi_b \) must coincide with \( \alpha_1 \), and again every point of \( \alpha_1 \) is singular.
with image in \( \alpha_1 \). On the other hand, if the image of a line \( l \) joining a general point on \( G_1 \) to its image on \( r_2 \) is a conic in an \( f \)-plane, then the \( f \)-planes associated with two such lines would have in common both \( S \) and its image \( S' \) and hence would have to coincide, which is impossible. Finally, if \( l \) is transformed into an entire \( f \)-plane or some other surface on \( V_2' \), then each of the \( \infty^3 \) \( f \)-planes on \( V_2' \) must meet the image of \( l \) in at least one point, and hence the image quadric of each \( f \)-plane must have a point in common with \( l \). Furthermore, in \( \alpha_1 \) there are \( \infty^1 \) lines joining a point on \( G_1 \) to its image on \( r_2 \). Hence, given a general \( f \)-plane, its image quadric, \( Q \), must have a curve in common with \( \alpha_1 \) and, since \( Q \) is not composite, this curve must be a straight line. Now since there exist \( \infty^2 \) \( \alpha_1 \)-planes, each of which is a \( b \)-plane on \( V_2' \), it follows that \( Q \) either contains \( \infty^2 \) lines or else contains \( \infty^1 \) lines each of which is common to \( \infty^1 \) \( b \)-planes. Since each of these is impossible, we conclude that every point of \( \alpha_1 \) is singular with image in \( \alpha_1 \). Hence the cubic complex of singular elements is composite, consisting of three linear complexes, at least two of which, \( \Gamma_{\alpha_1} \) and \( \Gamma_{\alpha_2} \), are special. Moreover, since each plane, \( \alpha_1 \), contains the corresponding \( E \)-point, \( G_1 \), and since in each \( f \)-plane the point \( G_1 \) is in \( \pi_1 \), it follows that \( \Gamma_{\alpha_1} \) contains the \( E \)-plane \( \pi_1 \). Similarly, \( \Gamma_{\alpha_2} \) contains the \( E \)-plane \( \pi_2 \). Hence the vertex, \( D_1 \), of \( \Gamma_{\alpha_1} \) lies in \( \pi_1 \) and the vertex, \( D_2 \), of \( \Gamma_{\alpha_2} \), lies in \( \pi_2 \).

The vertices \( D_1 \) and \( D_2 \) correspond to two lines, \( d_1 \) and \( d_2 \), in \( S_3 \) lying respectively in the planes, \( \tilde{\pi}_1 \) and \( \tilde{\pi}_2 \), which are represented on \( V_2' \) by the \( E \)-planes \( \pi_1 \) and \( \pi_2 \). Thus the complex \( \Gamma_{\alpha_1} \) consists of all lines meeting \( d_1 \), and the complex \( \Gamma_{\alpha_2} \) consists of all lines meeting \( d_2 \). Moreover, since the \( b \)-planes of both \( \Gamma_{\alpha_1} \) and \( \Gamma_{\alpha_2} \) are transformed into themselves, it follows that in \( S_3 \) a general bundle with vertex on either \( d_1 \) or \( d_2 \) is transformed into itself. Hence all lines in \( S_3 \) which meet both \( d_1 \) and \( d_2 \) are invariant, and on \( V_2' \) there is a nonsingular quadric surface of invariant points.

5. Conclusion. On \( V_2' \) let \( \pi_b \) be an arbitrary \( b \)-plane meeting \( \pi_1 \) in a line \( l \). Let \( A \) be a general point in \( \pi_b \), let its image be \( A' \), let \( \lambda \) be any line in \( \pi_b \) which contains \( A \), and let \( \pi_f \) be the \( f \)-plane which contains \( \lambda \). Then the intersection of \( \lambda \) and \( l \) is the point \( G_1 \) in \( \pi_f \). Hence \( \lambda \) is transformed into a generator, \( r_2 \), of the \( R_2 \) regulus on \( Q \), and therefore \( r_2 \) contains an \( E \)-point on the \( e_1 \)-component of the \( E \)-conic on \( Q \). Thus the image of \( \lambda \), which of course passes through \( A' \), meets \( \pi_1 \). Now the lines of \( V_2' \) which pass through \( A' \) and meet \( \pi_1 \) form a pencil lying in the \( b \)-plane which contains \( A' \) and meets \( \pi_1 \) in a line. Hence, considering all the lines of the pencil \( (\pi_b, A) \) it is clear that the proper image of
\( \pi_b \) is a second \( b \)-plane intersecting \( \pi_1 \) in a line. Thus in \( S_3 \), any bundle whose vertex is a point of \( \pi_1 \) has for its proper image another bundle whose vertex is also in \( \pi_1 \). A similar observation, of course, holds for any bundle whose vertex is a point of \( \pi_2 \).

We have thus shown that associated with any line involution possessing the properties described in the hypothesis of Theorem 1, there are two planes, \( \pi_1 \) and \( \pi_2 \), such that the line involution induces point involutions in each of these planes. Furthermore, since the order of the line involution is two, it is clear that the associated point involutions in \( \pi_1 \) and \( \pi_2 \) must be linear. Hence, since the points of the directrices, \( d_1 \) and \( d_2 \), of the two special linear complexes of singular lines are invariant under the respective point involutions, it follows that these point involutions must be harmonic homologies with \( d_1 \) and \( d_2 \) as their axes. The piercing points on \( \pi_1 \) and \( \pi_2 \) of the image, \( l' \), of a general line, \( l \), in \( S_3 \) are thus the images of the intercepts of \( l \) under harmonic homologies in \( \pi_1 \) and \( \pi_2 \), and the proof of Theorem 1 is complete.

References


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