INTERSECTIONS OF MAXIMAL STARSHAPED SETS

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0. Introduction. In Valentine [1, p. 183] the problem of characterizing starshaped sets in terms of maximal convex sets was posed. One published solution says that the convex kernel of a set is the intersection of all the maximal convex subsets of the set [2, p. 280]. In this paper we investigate the analogous problem of describing the intersection of all maximal starshaped subsets of a set. A maximal starshaped subset \( X \) of a set \( Y \) is a starshaped subset of \( Y \) which is not properly contained in any other starshaped subset of \( Y \). Since the property of being starshaped is not an intersectional property, it seems unlikely that the intersection of maximal starshaped subsets of a given set would be starshaped. Indeed, the following example shows the situation to be even more complex than merely absence of the intersectional property.

Let \( r^* = \{(x, y) | w - 1 \leq y \leq w, n - x \leq y \} \), and \( S^n = \bigcup_{i=1}^{n} T_i \); then \( S_n \) is starshaped with convex kernel, \( ck(S_n) \), equal to \( K_n = \{(x, y) | 0 \leq y \leq 1, n - x \leq y \} \). If \( S = \bigcup_{n=1}^{\infty} S_n \), then \( ck(S) \cap \bigcup_{n=1}^{\infty} ck(S_n) = \emptyset \). Thus \( S \) is not starshaped even though it is the union of an ascending chain of starshaped sets. Furthermore, \( S \) has no maximal starshaped subsets. If \( M \subseteq S \) were a maximal starshaped subset, then there would be at least one point \( (x, y) \in ck(M) \). In fact \( M \) would be precisely the set of points that \( (x, y) \) sees via \( S \). However, the point \((x + 1, y)\) sees every point which \( (x, y) \) does, and more. Thus \( M \) is not maximal.

In contrast with the preceding example, it is shown in §1 that compact subsets of Euclidean space, \( E^n \), have maximal starshaped subsets. In §2, it is shown that the intersection of the maximal starshaped subsets in a suitably restricted setting is starshaped.

1. Existence of maximal starshaped sets. Let \( S \) be a compact set in \( E^n \) and let \( \mathcal{F} \) denote the family of all classes \( C \) of maximal convex subsets of \( S \) for which \( \bigcap C \neq \emptyset \). Observe that a maximal convex subset of \( S \) is compact. We note two properties of \( \mathcal{F} \). First, if \( D \) is a finite subclass of some \( C \in \mathcal{F} \) then \( \emptyset \neq \bigcap C \subseteq \bigcap D \), so \( D \in \mathcal{F} \). Also, if \( C \) is a class of sets such that each finite subclass is in \( \mathcal{F} \), then \( C \in F \) by compactness and the definition of \( \mathcal{F} \). Thus \( \mathcal{F} \) is a family of finite character.

**Theorem 1.1.** There exists a maximal starshaped subset \( T \) of any compact set \( S \) in \( E^n \) and every maximal starshaped subset is closed.

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Proof. By the preceding, Tukey's Lemma gives a maximal class $C$ of maximal convex subsets of $S$ for which $\cap C \neq \emptyset$. Let $T = \cup C$; by using the fact that a starshaped set is the union of its maximal convex subsets, we see that $T$ is indeed a maximal starshaped subset of $S$. Noting that the closure, $T^-$, is starshaped and $T^- \subset S$, we see that $T = T^-$. That is, $T$ is closed.

Corollary 1.2. If $S \subset \mathbb{R}^n$ is compact and $B$ is any starshaped subset of $S$, then there exists a maximal starshaped subset $T \subset S$ such that $B \subset T$.

Proof. Express $B$ as the union of its maximal convex subsets; then let $T = \cup C$, where $C$ is one maximal class of maximal convex subsets of $S$ with $\cap C \neq \emptyset$, at least one member containing each one of the maximal convex subsets of $B$.

2. Intersections of maximal starshaped sets in the plane. Hereafter $S$ is always taken to be a compact simply connected set in the plane. Likewise $S_\alpha$, $\alpha$ in an index set $I$, will represent a maximal starshaped subset of $S$; and $A$ is taken to be the intersection of all the maximal starshaped subsets of $S$, i.e. $A = \cap_{\alpha \in I} S_\alpha$. We note that $A$, perhaps empty, is closed and thus compact.

Particular notations are as follows: $pq$ denotes the closed segment established by the points $p$ and $q$; $\Delta pqr$ denotes the convex hull of the three points, $p$, $q$, and $r$; $L(p, q)$ is the line established by the points $p$ and $q$; and $kC_{pq}$ denotes the cone opposite $p$ and $q$ with vertex $k$, i.e. $kC_{pq} = \{x : x = \lambda p + \mu q + \nu k, \lambda + \mu + \nu = 1, \lambda \leq 0, \mu \leq 0\}$.

Lemma 2.1. If $p, q \in A$, then $pq \subset A$ if and only if $pq \subset S$.

Proof. The "only if" part is immediate. If $pq \subset S$ and $k \in \text{ck}(S_\alpha)$, then we have $pq \cup \Delta pker \subset S$. So $\Delta pker \subset S$. This means that $k$ sees all of $\Delta pker$, so $S_\alpha \cup \Delta pker$ is a starshaped subset of $S$ having $k$ in its kernel. Consequently we have $pq \subset S_\alpha$ by the maximality of $S_\alpha$. That is $pq \subset A$.

Observe that simple connectedness and a standard sequence argument gives the following. If $p, q \in A$ and $pq \subset A$, then the set of points, $B$, that see $p$ and $q$ via $S$ is a compact set contained in one of the open half planes of $L(p, q)$.

Lemma 2.2. The set $B$, as given above, contains a unique element which is closest to $L(p, q)$.

Proof. Since the distance from points of $B$ to $L(p, q)$ is a positive continuous function defined on a compact set, we observe that a closest point exists. Distinct closest points $x$ and $y$ in $B$ establish
$L(x, y)$ parallel to $L(p, q)$. Adjusted notation, if necessary, gives $xq \cap yp$ to be a point of $B$ closer than the minimum distance.

**Lemma 2.3.** Every pair of points of $A$ can be joined in $A$ by a polygonal path of no more than two edges.

**Proof.** If $p, q \in A$ and $pq \subseteq A$, let $m$ be the unique point of Lemma 2.2. If $k \in \text{ck}(S_a)$, we have $k \subseteq mCpq$. But simple connectedness of $S$ gives the quadrilateral $kpmq$ and its interior to be a subset of $S$. Now $k$ sees $m$, so $pm \cup mq \subseteq S_a$. Since $\alpha$ was arbitrary, it follows that $pm \cup mq \subseteq A$.

**Theorem 2.4.** The intersection of the maximal starshaped subsets of a compact, simply connected set in $E_2$ is starshaped or empty.

**Proof.** Let $p, q, r$ be three points of $A$ such that no point of $A$ sees all three points via $A$. Otherwise Krasnoselskii’s Theorem says that $A$ is starshaped [1].

For the first case assume that $p, q,$ and $r$ are collinear with $q$ between $p$ and $r$.

By our initial assumption $pr \subseteq A$; suppose $qr \subseteq A$ and $pq \subseteq A$. Then Lemma 2.2 establishes a point $m$ closest to $L(q, r)$ and Theorem 2.3 yields $qm \cup mr \subseteq A$. As before $\text{ck}(S_a) \subseteq mCqr$ for any $\alpha$. If $k \in \text{ck}(S_a)$, we have $pk \cup kq \cup rk \subseteq S$. Simple connectedness gives $pm \subseteq S$. Thus, Lemma 2.1 says $pm \subseteq A$ and the resulting contradiction—$pm \cup mq \cup mr \subseteq A$—assures that $pq \subseteq A$. Similarly $qr \subseteq A$.

Let us now assume that none of the segments between $p, q,$ and $r$ is contained in $A$. Again Lemma 2.2 establishes a point $m$ closest to $L(p, r)$ for the points $p$ and $r$ with $pm \cup mr \subseteq A$. Select $k \in \text{ck}(S_a)$ and note that $k \subseteq mCpr$. If $m \in kq$, we have a contradiction, so assume that $kq \cap (rm \cup pm)$ is a point distinct from $m$. Without loss of generality let the point of intersection be on $rm$. Now apply Lemma 2.2 to establish a point $n$ closest to $L(p, q)$. The point $n$ must be such that $k \subseteq nCpq$, i.e. $n \in \Delta pkq$. Extend $qm$ to intersect $pk$ in a point $j$. If $n \in \Delta pqj$, we observe that $mq \subseteq A$, a contradiction. Otherwise, either $nr \subseteq A$ or $nq$ extended to intersect $mr$ yields a point of $A$ that sees $p, q$ and $r$ via $A$.

We now observe that all the above excludes the possibility of $p, q$ and $r$ being collinear.

Since $p, q$ and $r$ are not collinear, we employ them as a barycentric basis to describe regions of the plane. For example, a $(+, -, 0)$ point $k$ is such that $k = \alpha p + \beta q + \gamma r$ with $\alpha + \beta + \gamma = 1$ and $\alpha > 0$, $\beta < 0$, $\gamma = 0$. 

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Suppose \( k \in ck(S_\alpha) \); two cases are trivial—namely (0, 0, +) and (0, +, +). Particular permutations of these sign symbols are assumed without loss of generality. The three cases (0, −, +), (+, −, +) and (−, +, −) are disposed of simultaneously (i) with a proof identical in wording to the paragraph that dispenses with \( p, q \) and \( r \) being collinear and none of their segments in \( A \), and (ii) with minor modifications for the cases in which one of the segments \( pq, qr, pr \) is in \( A \).

The final possibility is for \( k \in ck(S_\alpha) \) to be a (+, +, +)-point. Here Lemma 2.2 and Theorem 2.3 assure us that \( kp \cup kq \cup kr \) is contained in a "three-pointed star region" all of whose edges are segments of \( A \). Simple connectedness and Lemma 2.1 ensure that this region (and, in particular \( kp \cup kq \cup kr \)) is a subset of \( A \).

References


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