THE ODD PRIMARY OBSTRUCTIONS TO FINDING
A SECTION IN A $V_n$ BUNDLE ARE ZERO$^1$

ARTHUR H. COPELAND, JR. AND MARK MAHOWALD

1. Let $\xi$ be a stable, orientable vector bundle over an $m$-dimensional space $X$. If $n$ is odd and $t < 2n - 1$, then $\pi_t(V_n)$ is 2-primary, as a simple application of the Serre $\mathcal{C}$-theory shows. As a result, finding a section of the associated $V_n$ bundle to $\xi$, for $m < 2n - 1$, depends only on the 2-primary cohomology of $X$. The purpose of this note is to prove the same result for $n$ even. Since $V_n$ is a mod $p$ $n$-sphere through dimension $2n - 1$ (when $n$ is even and $p$ is an odd prime), it is clear that the $p$-primary obstructions do not necessarily lie in zero groups. The precise statement of what we can prove requires us to introduce some notation. This is done in the next section.

The method of proof uses the idea of a modified Postnikov tower $^2$ and some of its more recent formulations.

2. It is possible to construct a resolution of $V_n$

$$
\begin{array}{cccc}
K(G_0) & K(G_1) & K(G_2) & \cdots \\
\uparrow & \uparrow & \uparrow & \\
V_n & V_1 & V_2 & \cdots
\end{array}
$$

with the following notation and properties. Each map $p^*$ is a fiber projection, with fiber $\Omega K(G_{s-1})$, induced by the vertical arrow on its left. When $G = \{G^s\}$ is a graded group, $K(G) = \prod_t K(G^t, t)$ is a product of Eilenberg-MacLane spaces. The graded groups are as follows,

$$
\begin{align*}
G_0^t &= H_t(V_n; Z) & t < 2n - 1 \\
&= 0 & t \geq 2n - 1 \\
G_1^t &= \sum_{p \neq 2} H_t(V_n^p; Z) \otimes Z_p & t < 2n - 1 & (1 \leq s < s_0) \\
&= 0 & t > 2n - 1 \\
G_2^t &= H_t(V_n^s; Z) \otimes Z_2 & t < 2n - 1 & (s_0 \leq s) \\
&= 0 & t \geq 2n - 1
\end{align*}
$$

Received by the editors July 20, 1967.

$^1$ This work was supported in part by a grant from the U. S. Army Research Office (Durham) and NSF Grant GP-5591. The second author is an Alfred P. Sloan fellow.
where \( s_0 \) is the smallest integer such that \( \sum_{t,p} H_t(V^n_{V'}; Z) \otimes Z_p = 0 \) (the sum is over \( t < 2n - 1 \) and \( p \) an odd prime). We require of this resolution that the cohomology homomorphisms induced by the fiber projections be trivial for certain choices of coefficient groups and in dimensions less than \( 2n - 1 \):

- \( p^{1*} = 0 \) with \( Z \) for coefficients;
- \( p^{**} = 0 \) with \( Z_p \) coefficients, \( p \) an odd prime, \( s < s_0 \);
- \( p^{**} = 0 \) with \( Z_2 \) coefficients, for \( s \geq s_0 \).

Note that \( s_0 = 1 \) when \( n \) is odd. When \( n \) is even, the existence of \( s_0 \) is insured by the fact that \( \pi_t(V^n) \) is finitely generated for each \( t \). Indeed, \( s_0 = n/2 \) if \( n \) is even. This fact also insures that the resolution has only finitely many terms in it.

Associated with such a resolution is a resolution of \( BSO_n \),

\[
B K(G_0) 
\begin{array}{c}
\downarrow f_n^0 \\
BSO \\
\end{array} 
\begin{array}{c}
\uparrow f_n^1 \\
E_n^1 \\
\end{array} 
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} 
\begin{array}{c}
\downarrow f_n^s \\
BSO_n \\
\end{array} 
\begin{array}{c}
\uparrow f_n^s \\
E_n^s \\
\end{array} 
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} 
\begin{array}{c}
\uparrow f_n^s \\
B K(G_0) \\
\end{array}
\]

where \( B K(G) = \prod_t K(G_t, t+1) \). Some details of how this can be done are given in [1].

The primary obstruction to showing that \( \xi \), a bundle over \( X \), has geometric dimension \( \leq n \) (i.e., to lifting \( \xi \) to \( \xi': X \to BSO_n \)) is the obstruction to lifting \( \xi \) to \( \xi_1: X \to E^1 \). The odd primary obstruction to lifting \( \xi \) to \( \xi' \) is the obstruction to lifting \( \xi_1 \) to \( \xi_{s_0}: X \to E^{s_0} \). Finally, the 2-primary obstruction is the obstruction to lifting \( \xi_{s_0} \) to \( \xi_\infty: X \to E^\infty \). Clearly \( BSO_n \to E^\infty \) is a \((2n-2)\)-homotopy equivalence. Thus if \( \dim X < 2n - 1 \), then the vanishing of these three kinds of obstructions is necessary and sufficient to produce a cross section.

**Theorem.** Let \( \xi \) be a stable orientable vector bundle over \( X \), a space of dimension less than \( 2n - 1 \), and suppose the primary obstruction to finding a section of the associated \( V_n \) bundle vanishes. Then the odd primary obstruction to finding such a section vanishes.

**Remark.** Although we have chosen a particular resolution in order to state our theorem, it is clear that this introduces no loss of generality. Indeed, the theorem shows that any obstruction theory which gives a complete solution to the cross section problem produces a resolution of the homotopy of \( V_n \) and, for any such resolution, all obstructions that are not primary but are in a \( p \)-primary group are zero.
First observe that if \( n \) is odd, the theorem is trivial. From now on we suppose \( n \) is even.

**Lemma.** If, in addition, the Stiefel-Whitney class \( w_n(\xi) = 0 \), then the conclusion holds.

**Proof.** There exists a natural diagram,

\[
\begin{array}{c}
E_1^1 \leftarrow \cdots \leftarrow E_n^s \leftarrow \cdots \leftarrow E_\infty^s \\
\xrightarrow{BSO} \downarrow \\
E_{n-1}^1
\end{array}
\]

Note that the space \( E_{n-1}^1 \) is the universal example for bundles with \( w_n = 0 \) and vanishing primary obstruction to finding a section of the associated \( V_n \) bundle. Thus the Lemma is proved if we show that \( E_{n-1}^1 \rightarrow E_n^s \) lifts to \( E_n^s \). Suppose we have \( \phi: E_{n-1}^1 \rightarrow E_n^s \). This gives

\[
\begin{array}{cccc}
E_{n-1}^1 & \phi & E_n^s \\
\uparrow & & \uparrow \\
BSO_{n-1} & BSO_n
\end{array}
\]

Now \( \ker (f_{n-1}^*) = 0 \) (mod \( p \)) in dimensions less than \( 2n-1 \). Also note that the \( k \)-invariants of \( E_{n+1}^s \rightarrow E_n^s \) are all in \( \ker (f_n^*) \), hence in \( \ker \phi^* \), whence \( \phi \) lifts to \( E_{n+1}^s \).

Now we can prove the theorem. Let \( \overline{X} \rightarrow X \) be the fibration induced by the map \( X \rightarrow K(Z_2, n) \) corresponding to \( w_n(\xi) \in H^n(X; Z_2) \). Then the Lemma gives the diagram,

\[
\begin{array}{c}
\overline{X} \rightarrow E_n^s \\
\downarrow \\
X \rightarrow E_n^1
\end{array}
\]

We can consider \( \overline{X} \subset X \), and the question is then just one of extending a lifting. The obstructions all lie in \( H^* (X, \overline{X}; \pi^*) \), where \( \pi^* \) is a group of odd order. Clearly \( H^* (X, \overline{X}; \pi^*) = 0 \), and this completes the proof.

**References**


Northwestern University