A CHARACTERIZATION OF INNER PRODUCT SPACES

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The well-known parallelogram law of Jordan and von Neumann [**1**] has been generalized in two different ways by M. M. Day [**2**] and E. R. Lorch [**3**], and both of these results are corollaries of the theorem to be proved here. Jordan and von Neumann established that in order for a normed linear space S to be an inner product space, it is necessary and sufficient that the following condition (JN) be satisfied: $\|P+Q\|^2+\|P-Q\|^2=2\|P\|^2+2\|Q\|^2$ for every P and Q in S. Day observed that (JN) could be restricted to points of norm one; i.e.,

CONDITION (D) $\|P+Q\|^2+\|P-Q\|^2=4$ for every P and Q on the unit sphere for S.[1]

Lorch's condition, instead of stipulating the specific functional relation (JN) between $\|P-Q\|$ and $\|P\|$, $\|Q\|$, and $\|P+Q\|$, requires simply that there exist some functional relation between them.

CONDITION (L) There exists some function F of three real variables such that $F(\|P\|, \|Q\|, \|P+Q\|)=\|P-Q\|$ for every P and Q in S.

The new condition (S) is (L) with the points restricted to the unit sphere: there exists some function F of one real variable such that $F(\|P+Q\|)=\|P-Q\|$ for every P and Q of norm one. It is interesting to note that (S) improves on (L) in the same way that (D) improves on (JN) and that (S) improves on (D) in the same way that (L) improves on (JN).

THEOREM. *Suppose that S is a normed linear space. The following two statements are equivalent*:

(1) *S is an inner product space, and*

(2) *There exists some real function F of one real variable defined over the number interval* $[0, 2]$ *such that if each of P and Q is a point of the unit sphere for S, then* $F(\|P+Q\|)=\|P-Q\|$.

PROOF. It is easily seen that statement (1) implies statement (2), for if S is an inner product space and P and Q are two points of the unit sphere for S, then $\|P+Q\|=(4-\|P-Q\|^2)^{1/2}$. Suppose that statement (2) is true. It may be assumed that S is two dimensional, for (JN) implies that a normed linear space is an inner product space if each two-dimensional subspace of it is an inner product space.

Presented to the Society, April 15, 1967; received by the editors May 9, 1967.

[1] Examination of Day's proof reveals that the relation "=" in (D) may be replaced either by "$\leqq$" or "$\geqq$", a fact later pointed out by Schoenberg [**4**].

To begin the proof, denote by M the unit sphere for S and let P be some point of M. There exists some point Q of M such that $\|P-Q\|=1$. Set up a coordinate system with P at $(1, 0)$ and Q at $(\cos(\pi/3), \sin(\pi/3))$. Denote by C the circle with radius 1 and center the origin. The argument that follows shows that M is C and therefore that S is an inner product space.

The two points P and Q are common to C and M. Since $|P-Q|=\|P-Q\|=1$, $P-Q$ belongs to both C and M. By the symmetry of M, $Q-P$, $-P$, and $-Q$ are also common to C and M. Therefore,

$$\begin{aligned}\|P - Q\| &= \|Q - (Q - P)\| = \|Q - P - (- P)\| = \| - P - (- Q)\| \\ &= \| - Q - (P - Q)\| = \|(P - Q) - P\| = 1.\end{aligned}$$

Statement (2) implies that on the unit sphere, the norm of the sum (difference) of two points determines the norm of their difference (sum). Therefore,

$$\|P+Q\| = \|2Q - P\| = \|Q - 2P\| = \| - P - Q\| = \|P - 2Q\| = \|2P - Q\|.$$

Denote $\|P+Q\|$ by c. The two points $2Q/c-P/c$ and $Q/c-2P/c$ belong to the unit sphere. The norm of their difference is 1 and the norm of their sum is $3/c$. This implies that $F(1)=3/c$. But since $\|P-Q\|=1$ and $\|P+Q\|=c$, $F(1)=c$. This means that $c=3/c$ and $c=(3)^{1/2}$. Since $|P+Q|=\|P+Q\|$, the point $(P+Q)/\|P+Q\|$ belongs to both C and M. By a similar argument, $(2Q-P)/\|2Q-P\|$ belongs to both C and M, and so forth, so that $\exp(2\pi in/12)$ belongs to both C and M for $n=1, 2, 3, \cdots, 12$. This is presently to be extended to 24 points, 48 points, etc., but the method used on these first 12 points cannot be carried on.

The number interval $[0, 2]$ is both the domain and the range of F. Also $F(0)=2$, $F(2)=0$, and $F(F(x))=x$ for each x in $[0, 2]$. Let H be the transformation from the number interval $[0, \pi]$ to the set F such that

$$H(\theta) = \left(\left\|P - \frac{P\cos(\theta) + Q\sin(\theta)}{\|P\cos(\theta) + Q\sin(\theta)\|}\right\|, \left\|P + \frac{P\cos(\theta) + Q\sin(\theta)}{\|P\cos(\theta) + Q\sin(\theta)\|}\right\|\right).$$

The transformation H is continuous, its domain is closed and compact, and its range is F. Therefore F is a closed and compact point set, and it follows that F is continuous. From this property it follows readily that F is decreasing over $[0, 2]$, for if $0\leqq x<y\leqq 2$ and $F(x)\leqq F(y)$, continuity and $F(2)=0$ imply that there is a y' such that

$x<y\leqq y'\leqq 2$ and $F(x)=F(y')$, which is a contradiction since F is its own inverse.

Suppose that n is a positive integer and that the common part of M and C has as a subset some point set K consisting of $6\cdot 2^n$ points evenly spaced about C. It has been shown that this supposition is valid for $n=1$ and it will now be shown valid for $n+1$, implying that the common part of M and C is dense in C and that M is C. Suppose that P_0, P_1, Q_0, and Q_1 are four points of K such that P_0 is adjacent to P_1 on C, Q_0 is adjacent to Q_1 on C, the point of C midway between P_0 and P_1 on the shorter arc of C from P_0 to P_1 is perpendicular to the point midway between Q_0 and Q_1 on the shorter arc of C from Q_0 to Q_1 and P_1 is between P_0 and Q_1 on the shorter arc of C from P_0 to Q_1. (That two points of K are adjacent to each other on C means that for some arc of C containing the two points, no point of K lies between them on that arc.) Rotate coordinates so that (1, 0) is halfway between P_0 and P_1 on the shorter arc of C. This puts both of Q_0 and Q_1 in either the upper half of the plane or in the lower half. The situation where (0, 1) is halfway between Q_0 and Q_1 on the shorter arc of C is considered in the sequel and the other situation could be treated in a similar manner.

Let P' denote $(P_0+P_1)/\|P_0+P_1\|$, and Q', $(Q_0+Q_1)/\|Q_0+Q_1\|$. It is first shown that $|P'|=|Q'|$. Let m_1 denote $|P'|$ and let m_2 denote $|Q'|$. Note that $\|P_0+P_1\|=|P_0+P_1|/m_1$ and that $\|Q_0+Q_1\|=|Q_0+Q_1|/m_2$. Suppose that $m_1\neq m_2$. The case $m_1>m_2$ is considered and a contradiction obtained. The other case, $m_2>m_1$, would yield a contradiction in a similar manner.

Let P_0' denote the point of K adjacent to P_0 and C and such that P_0 is between P_0' and P_1 on C. Denote by L the number $|P_0+P_1|/2$. Because M is the boundary of a convex point set, $|Z|\geqq L$ for each point Z of M. Denote $\pi/(6\cdot 2^n)$ by θ, and note that θ is half the angle between any two adjacent members of K. Denote by U the distance from the origin to the point on the X-axis that belongs to the line through P_0 and P_0'. Because of convexity, $|Z|\leqq U$ for each point Z of M. Let A and B denote $|P_0-P_1|$ and $|P_0+P_1|$ respectively. Let Q_2 be the member of K adjacent to Q_1 on C and such that Q_1 is between Q_0 and Q_2 on C. Denote by s the slope of the line through Q_1^* and Q_2^* where "$*$" indicates complex conjugate. The following relationships may be verified by elementary calculations.

$$L = \cos(\theta), \quad U = \cos(\theta)/\cos(2\theta),$$

$$A = |P_0 - P_1| = 2\sin(\theta), \quad B = |P_0 + P_1| = 2\cos(\theta),$$

$$|Q_2 - Q_2^*| = 2\cos(3\theta), \quad s = \sin(2\theta)/\cos(2\theta), \quad Q_1 = (A/2, B/2).$$

Since cosine is decreasing over $[0, \pi]$ and since θ and 3θ lie in this interval, $(\cos(\theta)+\cos(3\theta))/2>\cos(3\theta)$, so $\cos(\theta)\cos(2\theta)>\cos(3\theta)$, and $2\cos(2\theta)>2\cos(3\theta)/\cos(\theta)$. Now,

$$\begin{aligned}\|Q_1 - Q_1^*\| &= 2\cos(\theta)/m_2 > 2\cos(\theta)/m_1 = \|P_0 + P_1\| \\ &\geqq 2\cos(\theta)/U = 2\cos(2\theta) > 2\cos(3\theta)/\cos(\theta) \\ &= |Q_2 - Q_2^*|/L \geqq \|Q_2 - Q_2^*\|.\end{aligned}$$

Therefore, $\|Q_1-Q_1^*\|>\|P_0+P_1\|>\|Q_2-Q_2^*\|$, implying that between the two vertical chords $Q_1Q_1^*$ and $Q_2Q_2^*$ of M, there is some vertical chord R_1R_2 of M, R_1 between Q_1 and Q_2 on M, such that $\|R_1-R_2\|=\|P_0+P_1\|$. By the hypothesis, $\|R_1+R_2\|=\|P_0-P_1\|$. The following equations may now be written.

$$\begin{aligned}&\frac{F(\|P_0+P_1\|)-F(\|Q_0+Q_1\|)}{\|Q_0+Q_1\|-\|P_0+P_1\|} \\ &\qquad = \frac{\|P_0-P_1\|-\|Q_0-Q_1\|}{\|Q_0+Q_1\|-\|P_0+P_1\|} = \frac{A/m_2 - A/m_1}{B/m_2 - B/m_1} \\ &\qquad = \frac{A}{B} = \frac{\|R_1+R_2\|-\|Q_0-Q_1\|}{\|Q_0+Q_1\|-\|R_1-R_2\|} = \frac{\|R_1+R_2\| - A/m_1}{B/m_2 - |R_1-R_2|/m_2}.\end{aligned}$$

These equations are the beginning of a chain of inequalities which lead to a contradiction to the assumption that $m_1 \neq m_2$.

Let (x, r_1) be R_1 and let $(x, -r_2)$ be R_2 and suppose that $r_1 \geqq r_2$. (The case $r_2 \geqq r_1$ is just like this one and will not be considered.) Denote r_1-r_2 by d. The point Q_1^* is $(A/2, -B/2)$ and the line through Q_1^* and Q_2^* has slope s. It is evident that R_2 cannot be above this line and that the maximum difference between r_1 and r_2, leaving x fixed, occurs when R_1 and R_2 are each as high as possible. Since the line through Q_0 and Q_1 is parallel to the X-axis and since R_1 cannot be above this line, $r_1 \leqq B/2$. Similarly, $r_2 \geqq B/2 - s(x-A/2)$, so $d \leqq B/2 - (B/2 - s(x - A/2)) = s(x - A/2)$. Let r' denote $B/2 - s(x-A/2)$. This gives the following.

$$\begin{aligned}\frac{A}{B} &= \frac{\|R_1+R_2\| - A/m_1}{B/m_2 - |R_1-R_2|/m_2} = \frac{\|R_1+R_2\| - A/m_1}{B/m_2 - (r_1+r_2)/m_2} \\ &\geqq \frac{\|R_1+R_2\| - A/m_1}{B/m_2 - (2r' + d)/m_2}.\end{aligned}$$

Denote by L_1 the line through the origin and the point R_1+R_2. The slope s_1 of this line is $d/(2x)$, and since $d \leqq s(xA/2)$, $s_1 \leqq s(x-A/2)/(2x)$

$<s/2<s$. This makes it clear that L_1 is not as steep as the line L_2 through P_0 and the point $(m_1, 0)$ so L_1 and L_2 have in common some point Z. If L_2 is vertical, then $m_1=L$ and $m_2 \geqq m_1$, which is counter to the assumption made earlier that $m_1>m_2$. Therefore, L_2 is not vertical and has positive slope s_2. Note that $\|R_1+R_2\| \geqq |R_1+R_2|/|Z|$. Solving for $|Z|$, $|Z| = m_1s_2(4x^2+d^2)^{1/2}/(2s_2x-d)$, so $\|R_1+R_2\| \geqq (2s_2x-d)/(m_1s_2)$. Using this, the above inequality may be taken one step further.

$$\frac{A}{B} \geqq \frac{\|R_1+R_2\| - A/m_1}{B/m_2-(2r'+d)/m_2} \geqq \frac{m_2(2x-A-d/s_2)}{m_1(2x-A)s}.$$

Let s_2' denote the slope of the line through P_0' and P_0. It is evident that $s_2' \leqq s_2$. Since $s_2' = \cos(2\theta)/\sin(2\theta)$, it follows that $d/s_2 \leqq (x-A/2)s/s_2' = (x-A/2)\sin^2(2\theta)/\cos^2(2\theta) < x-A/2$. The chain of inequalities may now be terminated.

$$\frac{A}{B} \geqq \frac{m_2(2x-A-d/s_2)}{m_1(2x-A)s} \geqq \frac{m_2}{2m_1s} \geqq \frac{L}{2Us}$$

$$= \frac{\cos(\theta)}{2\cos(\theta)\sin(2\theta)/\cos^2(2\theta)} = \frac{\cos^2(2\theta)}{2\sin(2\theta)}.$$

Since $A=2\sin(\theta)$ and $B=2\cos(\theta)$, $\sin(\theta)/\cos(\theta) > \cos^2(2\theta)/2\sin(2\theta)$. By multiplying both sides by the product of the denominators and simplifying, one obtains $\cos(\theta) > 5\cos(3\theta)/2+\cos(5\theta)/2 > 5\cos(3\theta)/2 \geqq 5/(2(2)^{1/2}) > 1$. This is a contradiction caused by the assumption that $m_1 \neq m_2$ and proves that $m_1=m_2$.

For each integer k, let $H(k)$ mean the point $\exp(ik\theta)/\|\exp(ik\theta)\|$ and let $A(k)$ denote $|H(k)|$. All the points $H(k)$ belong to M, and if k is odd, $H(k)$ belongs to K and hence to C. Using the results just obtained, it is shown below that C contains all the points $H(k)$, implying that there exist $6\cdot 2^{n+1}$ points of M evenly spaced about C.

Since $\pi/2=3\cdot 2^n\theta$, the points $H(2k)$ and $H(2k+3\cdot 2^n)$ are related to each other as are P' and Q' in the above, so for every integer k, $A(2k) = A(2k+3\cdot 2^n)$. If k is an odd integer, both $H(k)$ and $H(k+3\cdot 2^n)$ belong to K, so $A(k)=A(k+3\cdot 2^n)$. This means that $A(k)=A(k+3\cdot 2^n)$ for every integer k. Suppose that j is an integer, $0 \leqq j \leqq n$, and, for each integer k, $A(k)=A(k+3\cdot 2^{n-j})$. This supposition is valid for $j=0$. In case $j<n$, it may be shown by the following argument that $A(k) = A(k+3\cdot 2^{n-(j+1)})$ for every integer k. Let X, Y, U, and V be as follows:

$$X = \|H(k) + H(k + 3\cdot 2^{n-j})\| = 2A(k)\cos(3\cdot 2^{n-j-1}\theta)/A(k + 3\cdot 2^{n-j-1});$$

$$\begin{aligned} Y &= \|H(k) - H(k + 3\cdot 2^{n-j})\| \\ &= 2A(k)\sin(3\cdot 2^{n-j-1}\theta)/A(k + 3\cdot 2^{n-j-1} + 3\cdot 2^{n}) \\ &= 2A(k)\sin(3\cdot 2^{n-j-1}\theta)/A(k + 3\cdot 2^{n-j-1}); \end{aligned}$$

$$\begin{aligned} U &= \|H(k + 3\cdot 2^{n-j-1}) + H(k + 3\cdot 2^{n-j-1} + 3\cdot 2^{n-j})\| \\ &= 2A(k + 3\cdot 2^{n-j-1})\cos(3\cdot 2^{n-j-1}\theta)/A(k); \end{aligned}$$

$$\begin{aligned} V &= \|H(k + 3\cdot 2^{n-j-1}) - H(k + 3\cdot 2^{n-j-1} + 3\cdot 2^{n-j})\| \\ &= 2A(k + 3\cdot 2^{n-j-1})\sin(3\cdot 2^{n-j-1}\theta)/A(k). \end{aligned}$$

Suppose that $X \neq U$. Then

$$\frac{F(X) - F(U)}{X - U} = \frac{Y - V}{X - U} = \frac{\sin(3\cdot 2^{n-j-1}\theta)}{\cos(3\cdot 2^{n-j-1}\theta)} > 0.$$

This contradicts the fact proved earlier that F is decreasing. Therefore, $X = U$ and this implies that for each integer k, $A(k) = A(k+3\cdot 2^{n-(j+1)})$. Therefore, by induction, $A(k) = A(k+3)$ for every integer k, and this puts all the points $H(k)$ on C. Thus M contains $6\cdot 2^{n+1}$ points evenly spaced about C.

It has now been shown that there exist $6\cdot 2^1$ points of M evenly spaced about C and that if there exist $6\cdot 2^n$ points of M evenly spaced on C, then there exist $6\cdot 2^{n+1}$ such points. Thus the common part of M and C is dense in C and M is C, completing the proof to the theorem.

This result is easily applied to a complex normed linear space. Associated with every complex normed linear space S_C, there is a real normed linear space S_R which has the same points as S_C, the same point addition, and the same meaning for multiplication of points by real numbers as in S_C. It is easily shown that S_C is a complex inner product space if and only if S_R is a real inner product space. If in S_C it is true that on the unit sphere the norm of the sum of two points determines the norm of their difference, then the same is true for S_R, implying that S_R is a real inner product space and hence that S_C is a complex inner product space.

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