A MOTZKIN-TYPE THEOREM FOR CLOSED NONCONVEX SETS

L. CALABI AND W. E. HARTNETT

Introduction. Bouligand [1] recognized the importance of the nearest-points mapping for a closed set $X$ and the set $S_X$ of points with more than one nearest point in $X$ for the study of geometry. Later Motzkin [3], [4] used them in the proof of his theorem characterizing closed convex sets. We use them to show, essentially, that $S_X$ characterizes the complement of $X$ in its convex hull. Our result includes the Motzkin theorem as a special case and yields a theorem of Valentine [5] as a corollary. The original motivation and background for our work can be found in [2].

The statement of the theorem. To every closed set $A$ of the Euclidean $n$-dimensional space $E$ we associate its closed convex hull $C(A)$ and its convex deficiency $D = D(A) = C(A) \setminus A$. We denote by $\pi$ the nearest-points mapping $A$ and by $r$ the distance from $A$:

$$r(x) = d(x, A), \quad \pi(x) = \{y: y \in A, d(x, y) = r(x)\},$$

where $d$ denotes the Euclidean distance.

We let $B(x)$ denote the closed ball around $x$ of radius $r(x)$ and $B^0(x)$ denote its interior. Observe that $B^0(x) \cap A = \emptyset$ and $B(x) \cap A = \pi x$. We shall say that $x \in A$ is a skeletal point of $A$ if $B(x)$ is contained in no other $B(x')$. The set of all skeletal points of $A$ is the skeleton of $A$ and is denoted by $S$. The skeletal pair of $A$ is $(S, q)$, where $q$ is the restriction of $r$ to $S$. Clearly $S$ contains all points having more than one nearest point in $A$; in fact, as already shown by Motzkin [3], such points form a dense subset of $S$.

Our main result may now be stated.

Theorem. Two closed subsets of $E$ have the same convex deficiency if and only if they have the same skeletal pair.

The proof of the theorem follows.

$D$ determines $(S, q)$. If $x, y \in E$ and $x \neq y$, we let $[x, y]$ denote the segment with endpoints $x$ and $y$ and set $[x, y] = [x, y] \setminus \{y\}$ and $(x, y] = [x, y] \setminus \{x\}$. We let $[y, x)$ denote the closed ray with endpoint

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y and set \( (y, x) = [y, x) \setminus \{y\} \). For \( y \in A \), \( \pi^-(y) = \{x : x \in E, y \in \pi x\} \).

For each set \( X \) we put \( X^* = \{x : x \in E, d(x, C(X)) = d(x, y) \) with \( y \in X\} \). Notice that \( X = C(X) \cap X^* \).

**Lemma 1.** If \( D \) is the convex deficiency of \( A \), we have:

(a) \( D^* \) is the complement of \( A^* \).

(b) \( A^* = A \cup \{x : x \in E, \) if \( y \neq x \) and \( y \in \pi x \), then \( \pi x = \{y\} \) and \[y, x) \subset \pi^-(y)\} \).

(c) \( D^* = \{x : x \in A, \) if \( y \in \pi x \), then \( \pi^-(y) \cap [y, x) = [y, z] \) for some \( z \} \).

**Proof.** Observe that \( x \in A^* \) iff \( d(x, C(A)) = d(x, A) \). Hence, because \( d(x, C(A)) \leq d(x, A), A^* = E \setminus D^* \) iff \( d(x, C(A)) < d(x, A) \) for each \( x \in D^* \).

If \( d(x, C(A)) < d(x, A) \), then \( d(x, C(A)) = d(x, y) \) for some \( y \in C(A) \setminus A = D \). Thus \( d(x, y) \leq d(x, C(A)) \leq d(x, C(D)) \leq d(x, y) \), since also \( y \in C(D) \). Consequently \( x \in D^* \).

Conversely, to prove that \( x \in D^* \) implies \( d(x, C(A)) < d(x, A) \), we prove that \( x \in D^* \) implies \( d(x, C(D)) = d(x, C(A)) = d(x, y) \) for some \( y \in D \). If \( x \in D \), that statement is trivial. Assume then \( x \in D^* \setminus D \), and hence also \( x \in C(D) \). Then, for some \( y \in D, d(x, C(D)) = d(x, y) \).

Let \( H_y \) be the hyperplane of support for \( C(D) \) at \( y \) orthogonal to \( [y, z] \) and let \( E_y \) be the closed half space bounded by \( H_y \) and containing \( C(D) \). If \( y' \in A \setminus E_y \), then \( [y, y'] \subset C(A) \) and, since \( D \subset E_y \), \( (y, y') \subset A \). But \( A \) is closed, and hence \( y \in A \), contradicting the fact that \( y \in D \). Then \( A \subset E_y \), \( C(A) \subset E_y \) and \( d(x, C(D)) = d(x, C(A)) = d(x, y) \) with \( y \in D \). If \( d(x, C(A)) = d(x, y) = d(x, A) \), then \( y \in A \) because \( A \) is closed. Hence \( d(x, C(A)) < d(x, A) \) and (a) is established.

To prove (b) it is enough to show that \( A^* \) contains the second set at the right of the equal sign so we pick \( x \) in the set. Then the hyperplane orthogonal to \( [y, x] \) passing through \( y \) is a hyperplane of support for \( A \) and hence for \( C(A) \). Thus \( y \in C(A) \) and \( d(x, A) = d(x, C(A)) \), that is \( x \in A^* \). Statement (c) follows at once from (a) and (b).

We set \( F(D) = (\text{bd } D) \setminus D \) and observe that \( D = \emptyset \) iff \( D^* = \emptyset \) iff \( F(D) = \emptyset \).

**Lemma 2.** If \( A \) has convex deficiency \( D \), then for \( x \in D^* \) we have \( r(x) = d(x, F(D)) \).

**Proof.** Suppose \( y \in \pi x \). If \( x \in D \subset C(A) \), then \( (y, x) \subset D, y \in D \), and hence \( y \in F(D) \). If \( x \in D^* \setminus D \), let \( y' = \pi_y x \) be the projection of \( x \) into \( C(D) \subset C(A) \). Then \( [y, y'] \subset C(A), (y, y') \subset C(A) \setminus A = D, y \notin D \), and so \( y \in F(D) \).

If \((S, q)\) is the skeletal pair of \( A \), we let \( P(A) = \bigcup \{(y, x) : y \in \pi x, x \in S\} \). We then have the following result:
Lemma 3. Suppose that $A$ has convex deficiency $D$ and skeletal pair $(S, q)$. Then $S \subset P(A) = D^*$.  

Proof. The inclusion is trivial. The equality follows from Lemma 1(c) and the observation that $x \in S$ iff $x \in A$ and $\pi^-(y) \cap [y, x] = [y, x]$ for $y \in \pi x$.

The proof of the next lemma is immediate.

Lemma 4. The skeleton $S$ of $A$ is the set of those points $x \in D^*$ for which

\[ r(x') + d(x, x') = r(x) \quad \text{if } x' \in [y, x], \]
\[ r(x) + d(x, x') > r(x') \quad \text{if } x' \in [y, x] \setminus [y, x] \]

for every $y \in \pi x$.

We can now establish the first half of the theorem. Let $A, A'$ be two closed sets with equal convex deficiency $D$. Then $P(A) = P(A')$ by Lemma 3, and $r(x) = r'(x)$ for each $x \in P(A)$ by Lemma 2. Lemma 4 yields $S = S'$ and consequently $q = q'$.

$(S, q)$ determines $D$. For each set $X$ we put $B^0(X) = \bigcup \{B^0(x) : x \in X\}$ and $\pi X = \bigcup \{\pi x : x \in X\}$. Notice that if $X \cap A = \emptyset$, then $B^0(X) \cap A = \emptyset$ and $\pi X \subset \text{bd } A$.

Lemma 5. Let $A$ have convex deficiency $D$ and skeletal pair $(S, q)$. Then

(a) $\pi x = B(x) \setminus B^0(S) \subset F(D)$ for each $x \in D^*$.

(b) $\text{Cl } \pi S = F(D)$.

Proof. First observe that $D \subset B^0(D^*) \subset B^0(S)$. Because $\pi x \subset B(x) \setminus B^0(S)$, it is enough to show that $B(x) \setminus B^0(S) \subset A$. If $x' \in (B(x) \setminus A) \cap C(A)$, then $x' \in D \subset B^0(S)$. Hence $(B(x) \setminus B^0(S)) \cap C(A) \subset A$. If $x' \in B(x) \setminus C(A)$, let $y \in \pi x$. Because $y \in C(A)$, $x' \in C(A)$, and $x \in A^*$, there exists a point $y' \in (y, x') \cap \text{bd } C(A) \subset B^0(x)$. Let $y_0 \in \text{bd } C(A) \cap B^0(x)$ be such that $d(x, \text{bd } C(A)) = d(x, y_0)$. Clearly, $y_0 \in D$ because $y_0 \in B^0(x)$. Let $H_{y_0}$ be a hyperplane of support for $C(A)$ and hence for $C(D)$ at $y_0$. There exists $z \in C(D)$ such that the open ray $(y_0, z)$ orthogonal to $H_{y_0}$ at $y_0$ lies in $D^* \setminus D$. Since, for $z' \in (y_0, z)$, $d(z', y_0) = d(z', C(D)) < d(z', A)$, it is easy to see that $x' \in B^0(z')$ for some $z' \in (y_0, z)$. But $B^0(z') \subset B^0(S)$ and hence $(B(x) \setminus B^0(S)) \cap C(A) = \emptyset$ and the equality $\pi x = B(x) \setminus B^0(S)$ is established. By Lemma 2, $\pi x \subset F(D)$ and (a) is proved.

From (a) we deduce $\text{Cl } \pi S \subset F(D)$, since the last set is closed. For the converse, if $y \in F(D)$, then $y \in \text{Cl } D$ and so there exists a sequence $\{x_n\}$ with $x_n \in D$ and $\{x_n\} \to y$. Let $y_n \in \pi x_n$; then $d(y, y_n) \leq d(y, x_n)$.
+d(x_n, y_n). Since \(\{x_n\} \to y\), \(r(x_n) \to 0\) and hence also \(d(y, y_n) \to 0\).
By Lemma 3, \(y_n \in \pi S\) and thus \(y \in \overline{\pi S}\).

**Lemma 6.** If \(D\) is the convex deficiency of \(A\), then \(D \subseteq C(F(D))\) and \(C(D) = C(F(D))\).

**Proof.** Suppose \(x \in D\) but \(x \notin C(F(D))\). Project \(x\) onto \(z \in C(F(D))\)
and let \(H_z\) be the supporting hyperplane of \(C(F(D))\) at \(z\) orthogonal
to \([z, x]\) and let \(E_z\) be the corresponding closed half space containing
\(C(F(D))\). We claim that \(C(A) \subseteq E_z\). If not, then there exists some \(y \in A \setminus E_z\). Because \(x \in C(A)\), \([x, y]\) lies in \(C(A)\). Now \([x, y] \cap A\) is
closed and nonempty and we let \(y'\) be the unique point of \([x, y] \cap A\)
nearest \(x\). Then \([x, y'] \subseteq D\) and \(y' \in F(D)\), contradicting the inclusions
\(F(D) \subseteq C(F(D)) \subseteq E_z\). Hence all points of \(A\) and of \(C(A)\) lie in \(E_z\),
in particular, \(x \in D \subseteq C(A)\), contradicting our assumption. Hence
\(D \subseteq C(F(D))\). But then \(C(D) \subseteq C(F(D)) \subseteq C(\overline{D}) = C(D)\) and the
second statement follows.

We can now terminate the proof of the theorem. Assume that \(A\)
and \(A'\) have same skeletal pair \((S, q)\). Then, by Lemma 5(a), \(\pi x = \pi' x\)
for each \(x \in S\) and consequently \(P(A) = P(A')\). Thus by Lemmas 3
and 5(b), \(D^* = D'^*\), \(F(D) = F(D')\). But by Lemma 6 then \(C(D) = C(F(D)) = C(D')\) and \(D = C(D) \cap D^* = D'\).

**References**

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Parke Mathematical Laboratories, Inc.