SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A SIMPLE EXISTENCE AND UNIQUENESS PROOF FOR A SINGULAR CAUCHY PROBLEM

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Consider the singular Cauchy problem in \( m + 1 \) space-time variables \( (x, t) = (x_1, x_2, \cdots, x_m, t) \) with \( u_i \) and \( u_{ij} \) denoting first and second space-derivatives,

\[
(1) \quad u_{tt} + kt^{-1}u_t = F(x, t, u, u_t, u_i, u_{it}, u_{ij}) \quad (t > 0),
\]

\[
(2) \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0.
\]

The functions \( F \) and \( f \) are analytic in their arguments.

**Theorem.** The singular Cauchy problem (1, 2) has for \( k \geq 0 \) a unique solution among the class of analytic functions.

Equation (1) is a generalization of the Euler-Poisson-Darboux (EPD) equation \( \Delta u = u_{tt} + kt^{-1}u_t \), whose study by A. Weinstein in 1952 [On the wave equation and the equation of Euler-Poisson, Amer. Math. Soc. Proc. 5th Sympos. Appl. Math., McGraw-Hill, New York, 1954, pp. 137–147] has given rise to literature too extensive to be listed here.

The proof makes use of the classic Cauchy-Kowalewski Theorem whose proof, when suitably modified, serves to establish this theorem also. Consider the formal power series

\[
(3) \quad u(x, t) = \sum c_{\alpha} x^\alpha t^\lambda \quad (\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)),
\]

where \( x^\alpha \) indicates the product of the \( x_i \)'s and their corresponding powers \( \alpha_i \); summation is over nonnegative powers. The initial data (2) and successive space-differentiations yield all coefficients of the form \( c_{\alpha 0} \) and \( c_{\alpha 1} \). Now write

\[
(4) \quad u_{tt} + kt^{-1}u_t = \sum \lambda(\lambda + k - 1)c_{\alpha} x^{\alpha} t^{\lambda - 2} = F(x, t, u, \cdots),
\]

and from this is determined \( c_{\alpha 2} \). Every \( c_{\alpha} \) can be expressed as a poly-

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nomial $P$ with positive coefficients in the $c_{\beta\mu}$ and the partial derivatives of $F$ (not denoting the dependency of $P$ on $\alpha$, $\lambda$)

$$\lambda + k - 1)c_{\alpha\lambda} = (\lambda - 1)P_{\alpha\lambda}(c_{\beta\mu}; F) \quad (\lambda > 1, \mu < \lambda).$$

If $k = 0$, then (1, 2) becomes a regular Cauchy problem, and (3) is, by the Cauchy-Kowalewski Theorem, convergent. Moreover, it is clear from (4) that the coefficients in the series solution of the regular equation are given by (5), with $k = 0$. An induction on $\lambda$ proves that $|c_{\alpha\lambda}|$ is no greater than the absolute value of the corresponding coefficient in the case $k = 0$. Hence the series (3) is absolutely and uniformly convergent for $k \geq 0$, and represents the solution of singular problem (1, 2).

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