ON RECONSTRUCTING A GRAPH

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1. Introduction. The term "graph" will here denote an unoriented finite graph without loops or multiple edges. $V(G)$ will denote the vertex set of $G$ and $E(G)$ will denote the edge set. If $a \in V(G)$, we will let $G_a$ denote the graph obtained from $G$ by deleting the vertex $a$ and the edges adjacent to $a$. If $e \in E(G)$ we will let $G^e$ denote the graph obtained from $G$ by deleting $e$. P. J. Kelly [3] has proven the following theorem: If $G$ and $H$ are trees and $\sigma: V(G) \to V(H)$ is a 1-1 onto function such that $G_a \cong H_{\sigma(a)}$ for all $a \in V(G)$, then $G \cong H$. He conjectures that this theorem is true for arbitrary graphs and has verified it for graphs on $n$ vertices where $2 < n \leq 6$. An equivalent formulation of Kelly's conjecture is as follows: $G$ is uniquely determined, up to isomorphism, by the collection $\{G_a\}_{a \in V(G)}$. We will refer to this as the vertex problem. If a graph $G$ is uniquely determined, up to isomorphism, by a given collection of subgraphs we will say that $G$ can be reconstructed from that collection of subgraphs. It needs to be emphasized that the given subgraphs have no labellings.

Harary and Palmer [1] generalized Kelly's theorem on trees by showing that a tree $G$ can be reconstructed from the $G_a$ with $a$ of degree one in $G$.

In [2], Harary asks if $G$ can be reconstructed from the collection $\{G^e\}_{e \in E(G)}$. We will refer to this as the edge problem. The purpose of this paper is to show that the edge problem is a special case of the vertex problem.

Undefined terms in the paper can be found in the above-mentioned papers or in [4].

2. The use of the line graph. If $G$ is a graph, then the line graph of $G$, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and with $(e_1, e_2) \in E(L(G))$ if and only if $e_1$ and $e_2$ are adjacent in $G$.

**Lemma.** Let $G$ be a given graph. Then $L(G^e) = (L(G))_e$ for all $e \in E(G)$.

**Proof.** Both graphs have $E(G) - \{e\}$ as vertex set, and if $e_1, e_2 \in E(G) - \{e\}$, then the criterion for $(e_1, e_2)$ to be an edge in either graph is the same; namely that $e_1$ and $e_2$ be adjacent in $G$.

Since the number of isolated vertices in $G$ can be discovered from the $\{G^e\}_{e \in E(G)}$ we assume in the following that $G$ and $H$ have no isolated vertices.

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Theorem. The edge problem is equivalent to the vertex problem for line graphs; i.e., a solution to the edge problem would give a solution to the vertex problem for line graphs and conversely.

Proof. Suppose the vertex problem is true for line graphs. Let $G$ and $H$ be graphs and let $\tau: E(G) \rightarrow E(H)$ be a 1-1 onto function such that $G^e \cong H^\tau(e)$ for all $e \in E(G)$. By the Lemma we then have $(L(G))_e = L(G^e) \cong L(H^\tau(e)) = (L(H))_{\tau(e)}$ for all $e \in E(G)$. But then $\tau: V(L(G)) \rightarrow V(L(H))$ is a 1-1 onto function such that $(L(G))_e \cong (L(H))_{\tau(e)}$ for all $e \in V(L(G))$ so by our assumption $L(G) \cong L(H)$. Now $G$ and $L(G)$ have the same number of components so $G$ and $H$ have the same number of components and by Whitney's Theorem [5], or see pp. 248 of [4] $G$ and $H$ have the same number of components of each isomorphism type with the possible exception of triangles and 3-pointed stars.

If for each $e \in E(G)$, $e$ is from a triangle component of $G$ if and only if $\tau(e)$ is from a triangle component of $H$, then $G \cong H$ since they would have the same number of triangle components. If there is some $e \in E(G)$ such that $e$ is from a triangle component but $\tau(e)$ is not then $\tau(e)$ must be from a 3-pointed star component of $H$. But then $G^e \not\cong H^\tau(e)$ since the latter has one more component than the former. (Removing $\tau(e)$ from the star leaves a path of length two and an isolated vertex.) One gets the same contradiction if $e$ is not from a triangle component while $\tau(e)$ is.

The proof that the vertex problem for line graphs is valid if the edge problem is valid is omitted because of its similarity to the above proof.

Corollary. If $G$ is disconnected then $G$ can be reconstructed from the collection $\{G^e\}_{e \in E(G)}$.

Proof. $L(G)$ can be constructed from the collection $(L(G))_e$ by [2] since $L(G)$ is disconnected.

It should be pointed out that one can decide from the $G^e$ if $G$ is connected or not. This follows from the observation that $G$ is connected if and only if either $G^e$ is connected for some $e \in E(G)$, $G^e$ is a forest with exactly two trees for all $e \in E(G)$ and for some $e \in E(G)$ neither component of $G^e$ is an isolated vertex, or else $G^e$ is a star plus an isolated vertex for each $e \in E(G)$.

References


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