TENSOR PRODUCTS OF SIMPLE PURE INSEPARABLE FIELD EXTENSIONS

J. N. MORDESON AND B. VINOGRADE

Let $K$ be a field of characteristic $p \neq 0$ and let $L$ be a pure inseparable extension field of finite degree over $K$. Our purpose is to give several necessary and sufficient conditions for $L$ to be a tensor product of simple extensions over $K$. Weisfeld [4] has a criterion, namely the existence of a nontrivial higher derivation of $L$ with $K$ as its subfield of constants, (in fact Weisfeld proves his criterion for infinite extensions of bounded exponent). The present note describes different criteria, in terms of Pickert's canonical generators [3, p. 133]. For a given canonical generating set $\{b_1, \ldots, b_r\}$ of $L$ over $K$, let $M_i = K(b_1, \ldots, b_i)$ and let $q_i$ denote $p^{e_i}$ where $e_i$ is the exponent of $b_i$ over $M_{i-1}$, $i = 1, \ldots, r$, where $M_0 = K$. We shall prove the following theorem.

**Theorem.** If $L$ is a finite degree pure inseparable extension of $K$, then the following conditions are equivalent:

1. $L$ is the tensor product of a finite number of simple extensions with respect to $K$.
2. Every canonical generating set is such that $b_i \in (L^{q_i} \cap K)(b_1, \ldots, b_{i-1}) = M_{i-1}^{q_i}(L^{q_i} \cap K)$, $i = 1, \ldots, r$.
3. Every canonical generating set is such that the tensor product $L \otimes M_i$ with respect to $K$ cleaves over $1 \otimes M_i$ (that is, $L \otimes M_i$ has a Wedderburn factor as an algebra over $1 \otimes M_i$), $i = 1, \ldots, r$.
4. There exists a canonical generating set such that $L \otimes M_i$ cleaves over $1 \otimes M_i$, $i = 1, \ldots, r$.
5. There exists a canonical generating set such that $b_i \in M_i^{q_i}(L^{q_i} \cap K)$, $i = 1, \ldots, r$.

**Proof.** (0) implies (1): Suppose $L \cong K(a_1) \otimes \cdots \otimes K(a_r)$ and that $\{a_1, \ldots, a_r\}$ is already ordered so that it is a canonical generating set. Let $\{b_1, \ldots, b_r\}$ be any given canonical generating set. For any $c \in L$, $c^{q_i} = (\sum j, k, a_1^{q_i} \cdots a_r^{q_i})$, where $k_j \in K$ and $j = \{j_1, \ldots, j_r\}$. By the division algorithm, $a_n^{j_n q_i} = a_n^{j_n r_n} a_n^{j_n - r_n}$ where $0 \leq r_n < q_n$ ($n = 1, \ldots, i-1$). Since $q_i$ divides $q_n$, $r_n$ has the form $q_n$ ($n = 1, \ldots, i-1$). Thus, since $\{a_1^{q_i}, \ldots, a_r^{q_i}\} \subseteq K$ and $\{a_1^{q_i}, \ldots, a_r^{q_i-1}\} \subseteq L^{q_i} \cap K$, there

---

Received by the editors September 8, 1967.

1 This paper was partially supported by NSF Grant G-23418.
exists a set \( \{ k'_t \} \subseteq L^t \cap K \) such that for \( t = \{ t_1, \ldots, t_{i-1} \} \),
\[
(*) \quad c^{q_i} = \sum_t k'_t a_{t_1^{q_i}} \cdots a_{t_{i-1}^{q_i}} , \quad 0 \leq q_{d_n} < q_n \ (n = 1, \ldots, i-1).
\]
Since the monomials \( \{ a_{t_1^{q_i}} \cdots a_{t_{i-1}^{q_i}} \} \) are linearly independent over \( K \), this set \( \{ k'_t \} \) is the only subset of \( K \) satisfying \((*)\). In particular,
\[
(**) \quad b_i^{q_i} = \sum_t k'_t a_{t_1^{q_i}} \cdots a_{t_{i-1}^{q_i}} , \quad k_t \in L^t \cap K , \quad t = \{ t_1, \ldots, t_{i-1} \}
\]
and \( 0 \leq q_{d_n} < q_n \ (n = 1, \ldots, i-1) \). Also,
\[
b_i^{q_i} = \sum_s k''_s b_1^{q_t} \cdots b_{t-1}^{q_{t-1}}, \quad k''_s \in K , \quad s = \{ s_1, \ldots, s_{i-1} \}
\]
and
\[
0 \leq q_{s_t} < q_n \quad (n = 1, \ldots, i-1).
\]
Thus, by \((*)\),
\[
b_i^{q_i} = \sum_s k''_s (b_1^{q_t} \cdots b_{t-1}^{q_{t-1}})^{q_i}
\]
\[
(***) \quad = \sum_s k''_s \left( \sum_t k'_t a_{t_1^{q_t}} \cdots a_{t_{i-1}^{q_t}} \right),
\]
k\( s_t \in L^s \cap K \) and \( 0 \leq q_{s_t} < q_n \ (n = 1, \ldots, i-1) \). Therefore, by \((**)\) and \((***)\), \( k_t = \sum_s k''_s k_{st} \) for each \( t \). Since the set \( \{ k''_s \} \) exists, the system \( k_t = \sum_s x_s k_{st} \) has a solution in \( L^s \cap K \), say \( x_s = k''_s \in L^s \cap K \). Hence,
\[
b_i^{q_i} = \sum_s \left( \sum_t k''_s k_{st} \right) a_{t_1^{q_t}} \cdots a_{t_{i-1}^{q_t}}
\]
\[
= \sum_s k''_s \left( \sum_t k'_t a_{t_1^{q_t}} \cdots a_{t_{i-1}^{q_t}} \right)
\]
\[
= \sum_s k''_s (b_1^{q_t} \cdots b_{t-1}^{q_{t-1}})^{q_i} \in M_{t-1}^i(L^t \cap K).
\]

\( (4) \) implies \((0)\): Make the induction hypothesis that \( L \cong M_i \otimes M_i' \) where \( M_i' = K(a_{t_i+1}) \otimes \cdots \otimes K(a_r) \) (there being nothing to prove for \( i = r \)). Since \( b_i^{q_i} = \sum_j \sum j_1^{q_j} b_{j_1}^{q_j} \cdots b_{t_i}^{q_j} \) where \( k_j = c_j^{q_j} \in L^t \cap K \) and \( j = \{ j_1, \ldots, j_{i-1} \} \), we have \( b_i = \sum_j c_j b_{j_1}^{q_j} \cdots b_{t_i}^{q_j} \). Hence, \( M_i = M_{i-1}(b_i) = M_{i-1}(\{ c_j \}) \). Since \( M_i \) is simple pure inseparable over \( M_{i-1} \), there exists \( a_i \in \{ c_j \} \) such that \( M_i = M_{i-1}(a_i) \) and \( a_i \in K \). Since \( [M_i : K] = q_1 \cdots q_i \), it follows that \( [M_{i-1}(a_i) : M_{i-1}] = q_i \).
Hence, \( M_i \cong M_{i-1} \otimes K(a_i) \). Thus, \( L \cong M_{i-1} \otimes M'_{i-1} \) where \( M'_{i-1} = K(a_i) \otimes \cdots \otimes K(a_r) \). Hence, by induction, \( L \cong K(a_1) \otimes \cdots \otimes K(a_r) \). (E. A. Hamann has a different proof of this implication.)

Since (1) implies (4) trivially, we have the equivalence of (0), (1) and (4).

(1) implies (2): Since \( \beta_i \in M_{i-1}^d(\mathbb{L} \cap K) \), \( i = 1, \ldots, r \), we have \( \beta_i = \sum_j c_{i,j} m_j \) where \( c_{i,j} \in L \), \( m_j \in M_{i-1} \) and \( c_{i,j}^* = k_j \in K \). Let \( \beta'_i = \sum_j c_{i,j} \otimes m_j \). Then \( \beta'^*_i = \sum_j c_{j}^* \otimes m_j^* = \sum_j 1 \otimes k_j m_j^* \in 1 \otimes M_{i-1} \). Since \( e_i \) is the exponent of \( \beta_i \) over \( M_{i-1} \), \( M'_i = (1 \otimes M_{i-1})[\beta'_i] \) is a field \( (i = 1, \ldots, r) \).

Now consider \( L \otimes M_i \) for any \( i = 1, \ldots, r \). Suppose there exists a field \( M_i \) such that \( M_i \supseteq 1 \otimes M_i \) and \( f_i : M_i \to M_i \) where \( f_i \) is the canonical \( K \)-epimorphism of \( L \otimes M_i \) onto \( L \otimes M_i = L \). By the previous paragraph, there exists a field \( M'_i \subseteq L \otimes M_i \) such that \( M'_i \supseteq 1 \otimes M_i \) and \( f_i : M_i \to M_i \). Thus, there exists a field \( M'_i \) in \( L \otimes M_i \) such that \( M'_i \supseteq 1 \otimes M_i \) and \( f_i : M'_i \to M_i \). By the universal mapping theorem for tensor products, there exists a \( K \)-epimorphism \( h_i \) of \( L \otimes M_i \) onto the ring composite \( [L \otimes 1, M'_i] \otimes L \otimes M_i \) such that \( f_i = h_i M_i \). Hence, the proof follows by induction.

(2) implies (3): Immediate.

(3) implies (4): Let \( \{ b_1, \ldots, b_r \} \) be any canonical generating set such that \( L \otimes M_i \) cleaves over \( 1 \otimes M_i \), \( i = 1, \ldots, r \). Use the symbol \( \otimes_1 \) to denote the tensor product with respect to \( M_1 \). Then there is a canonical \( K \)-epimorphism of \( L \otimes M_i \) onto \( L \otimes M_i \), whence \( L \otimes M_i \) cleaves over \( 1 \otimes M_i \), \( i = 1, \ldots, r \). Now make the induction hypothesis that (3) implies (4) for all pure inseparable extensions of multiplicity less than \( r \). (3) implies (4) trivially for \( r = 1 \). Then since we have proved (4) is equivalent to (0), \( L \cong M_1(b_1') \otimes_1 \cdots \otimes_1 M_1(b_r') \) and we may assume \( \{ b'_1, \ldots, b'_r \} \) is canonically ordered over \( M_1 \). Since \( b_1 \) has maximal exponent in \( L \) over \( K \) and \( b'_1 \) has maximal exponent in \( L \) over \( M'_{i-1} \) \( (M'_i = M_i = K(b_1') \) and \( M'_j = M_i(b'_j, \ldots, b'_r), \)
\( j = 2, \ldots, r) \), it follows that \( \{ b_1, b'_2, \ldots, b'_r \} \) is a canonical generating set of \( L \) over \( K \), whence \( r = r' \). In particular,

\[ b'_j \otimes_{i-1} K(b_1', b_2', \ldots, b_{j-1}') \cap M_1, \quad j = 2, \ldots, r, \]

since the \( e_i \) of a canonical generating set are invariant. Because \( L \otimes M_1 \) cleaves over \( 1 \otimes M_1 \), there exists \( b'_j \otimes_1 L \otimes M_1 \) such that \( f_i b'_j = b'_j \) and \( b'^*_j \otimes_1 1 \otimes M_1 \), \( j = 2, \ldots, r \). Now \( b'_j = \sum c_{i,j} \otimes b'_i, \ c_i \in L \), whence \( b'^*_j = \sum c_{i,j} \otimes b'_i \). By the division algorithm, \( b_1 q_{1} = b_1 q_{1} = b_1 \), \( q_{1} \) divides \( q_{1} \), it follows that
Examples where \( L \) is not a tensor product of simple extensions can be found in [1, Ex. 6, p. 196] and [2, p. 51]. If \( \{ b_1, \ldots, b_r \} \) is a canonical generating set satisfying (4), it does not follow that \( L \cong K(b_1) \otimes K(b_r) \). For example, consider a perfect field \( P \) and independent indeterminates \( s, t \) over \( P \). Let \( K = P(s, t)(s^{1/p} + t^{1/p}) \) and \( L = P(s^{1/p^2}, t^{1/p^2}) \). If \( b_1 = s^{1/p^2} \) and \( b_2 = t^{1/p^2} \), then \( \{ b_1, b_2 \} \) is a canonical generating set with \( e_1 = 2 \) and \( e_2 = 1 \). It is easily verified that \( b_2^{p^2} \in (L^{p^2} \cap K)(b_2^{p^2}) \), but \( L \not\cong K(b_1) \otimes K(b_2) \) since \( b_2 \) has exponent 2 over \( K \). However, \( L \cong K(s^{1/p^2}) \otimes K(s^{1/p^2} + t^{1/p^2}) \).

The extent to which these results are valid in arbitrary pure inseparable extensions is considered by the authors in an article to appear in the Mathematische Zeitschrift. Other recent results can be found in an article by Haddix and Mordeson in the Formosan Science and in an article by Sweedler in the Annals. The equivalence of (0) above and the linear disjointness of \( K \) and \( L^{p^i} \) \((i = 1, 2, \ldots)\) is proved independently in these articles.

**References**


Creighton University and Iowa State University