ON QUASI-LOCAL NOETHERIAN RINGS

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It is the purpose of this note to show that each semiprime, quasi-local, noetherian ring with gl. dim $R \leq 2$ is Morita equivalent to a quasi-local noetherian domain $D$ with gl. dim $D \leq 2$ (cf. Theorem 1).

All rings considered here have an identity element; all modules are assumed to be unitary. The ring $R$ is noetherian if $R$ satisfies the ascending chain conditions for right and for left ideals. A domain is a ring without zero-divisors $z \neq 0$. The ring $R$ is quasi-local, if its Jacobson radical $J$ is its unique maximal two-sided ideal.

Our result here is another consequence of the Morita Theorems (cf. Auslander and Goldman [1, Appendix]). In order to apply them we need the following standard notation:

If $P$ is a finitely generated right $R$-module, and $T = \text{End}_R(P)$, then $P$ is also a left $T$-module. The map

$$\tau: \text{Hom}_R(P, R) \otimes_T P \to R$$

which is defined by $\tau(f \otimes x) = f(x)$ for all $x \in P$ and all $f \in \text{Hom}_R(P, R)$ is called the trace mapping of the $R$-module $P$. The image $\tau_R(P)$ of $\tau$ is the trace ideal of $P$. One statement of the Morita Theorems is that $\tau_R(P)$ is an idempotent, two-sided ideal of $R$, if $P$ is a finitely generated projective right $R$-module. In case we also have $\tau(P) = R$, then $P$ is a finitely generated projective left $T$-module, and $R \cong \text{End}_T(P)$.

Theorem 1. The ring $R$ is a semiprime, quasi-local, noetherian ring with gl. dim $R \leq 2$ if and only if $R$ is isomorphic to the full ring of endomorphisms $\text{End}_D(P)$ of a finitely generated projective left $D$-module $P$ over a quasi-local, noetherian domain with gl. dim $D \leq 2$.

Proof. If $R$ is a semiprime noetherian ring, then $R$ has a uniform right annihilator $P \neq 0$ by Goldie [2, p. 205, Theorem 2.3]. Hence $P = \tau = \{x \in R \mid tx = 0\}$ for some $0 \neq t \in R$ by Goldie [2, p. 208, Theorem 3.7]. Since gl. dim $R \leq 2$, the following standard exact sequence

$$0 \leftarrow R/\tau R \leftarrow R \leftarrow tR \leftarrow t \tau = P \leftarrow 0$$

shows that $P$ is a projective right $R$-module, which is finitely gen-
erated. Thus \(0 \neq \tau_R(P) = S\) is an idempotent ideal of \(R\). Since \(R\) is quasi-local, either \(S \leq J\) or \(S = R\) where \(J\) is the Jacobson radical of \(R\). Therefore \(S = SJ = 0\) by Nakayama’s Lemma, in case \(S \leq J\). This implies \(S = R\), and so \(R \cong \text{End}_D(P)\) by the above remarks, where \(D = \text{End}_R(P)\). Furthermore, \(P\) is a finitely generated projective left \(D\)-module. If the right \(R\)-module \(P\) is generated by \(n\) elements, then \(P\) is a direct summand of a free right \(R\)-module \(F\) on \(n\) generators.

Let \(B = \text{End}_D(F)\). Then there is an idempotent \(0 \neq e \in B\) such that \(D \cong e B e\). Clearly, \(B\) is quasi-local and noetherian, and \(\text{gl. dim } B \leq 2\) by Harada [3, Theorem 2]. Hence \(B e B = B\), which implies again by the Morita Theorems that \(eB\) is a projective left \(e B e\)-module. Thus \(\text{gl. dim } D = \text{gl. dim } (e B e) \leq \text{gl. dim } B \leq 2\) by Harada [3, p. 27, Theorem 8]. Obviously \(D\) is quasi-local and noetherian. Since \(D\) is the full ring of \(R\)-endomorphisms of the uniform right ideal \(P\) of \(R\), \(D\) is a domain by Goldie [2, p. 218, Theorem 5.6]. This completes the proof of Theorem 1, because the converse part is now obvious.

**Corollary 1.** A semiprime, quasi-local, noetherian ring \(R\) with \(\text{gl. dim } R \leq 2\) is a prime ring.

The proof follows at once from Theorem 1, because it states that \(R\) is Morita equivalent to a domain.

**Remark.** We do not know whether Theorem 1 holds, if we drop the requirement that \(R\) be semiprime, but assume that \(R\) has an artinian total ring of quotients. Since hereditary, quasi-local, noetherian rings are prime rings, one could expect an affirmative answer to this question. For quasi-local noetherian rings with \(\text{gl. dim } R \leq 2\) whose Jacobson radical is a principal right ideal we can show that they are prime rings, because the following statement holds.

**Corollary 2.** If \(R\) is a quasi-local noetherian ring with \(\text{gl. dim } R \leq 2\) whose Jacobson radical is a principal right ideal of \(R\), then \(R\) is Morita equivalent to either a simple noetherian domain or to a quasi-local noetherian domain \(D\) with \(\text{gl. dim } D \leq 2\).

**Proof.** Since \(R\) is a quasi-local noetherian ring whose Jacobson radical \(J\) is a principal right ideal of \(R\), the ring \(R\) is either semiprime or \(J\) is nilpotent by [4, Hilfssatz 4.1]. By Theorem 1 we may assume that \(J\) is nilpotent. If \(J = nR\), then let \(P = n_1 = \{x \in R \mid xn = 0\}\). Since \(\text{gl. dim } R \leq 2\), \(P\) is a finitely generated projective left \(R\)-module. Therefore \(\tau_R(P) = R\), because \(R\) is quasi-local. Hence

\[1 = \sum \phi_i f_i + \cdots + \phi_n f_n\]

for some \(\phi_i \in P\) and some \(f_i \in \text{Hom}_R(P, R)\). If we had \(P \leq J\), then there
would be a smallest positive integer \( k \) such that \( P^k = 0 \). Since \( k \neq 1 \), there exists \( 0 \neq y \in P^{k-1} \). Now

\[
y = y(p_1f_1) + y(p_2f_2) + \cdots + y(p_nf_n)
\]

\[
= (yp_1)f_1 + (yp_2)f_2 + \cdots + (yp_n)f_n = 0,
\]

because \( yp_i \in P^k = 0 \) for all \( i \). This contradiction shows \( P \nsubseteq J \). Hence \( R = PR \) by Nakayama’s Lemma. Thus

\[
R^n = PRn \subseteq P(nR) = (Pn)R = 0.
\]

Therefore \( J = 0 \), which implies that \( R \) is simple. Hence \( R \) is Morita equivalent to a simple noetherian domain \( D \) with \( \text{gl. dim } R \leq 2 \) by Theorem 2, completing the proof of Corollary 2.

References


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