A REMARK ON A THEOREM OF A. WEIL
MORIKUNI GOTO

1. The purpose of this paper is to prove the following theorem.

**Theorem.** Let $G$ be a connected semisimple Lie group without compact components. Let $H$ be a subgroup of $G$ such that there exists a compact subset $K$ of $G$ with $G = HK$. Let $\sigma$ be a continuous automorphism of $G$ which reduces to the identity on $H$. Then $\sigma$ is the identity automorphism of $G$.

This is a generalization of a theorem of A. Weil in [4], and is related to a theorem of A. Borel in [1]. The author obtained the theorem by globalizing the infinitesimal method of Weil in [4].

2. Let $G$ be a connected semisimple Lie group. Let $A(G)$ be the group of all continuous automorphisms of $G$. $A(G)$ is a Lie group with respect to the compact-open topology. Let $I(G)$ be the subgroup of $A(G)$ composed of all inner automorphisms. Then $I(G)$ is a closed normal subgroup of finite index in $A(G)$. For $g$ in $G$ we define $\text{Ad}(g)$ by $\text{Ad}(g)h = ghg^{-1}$ ($h \in G$). Then $G \ni g \mapsto \text{Ad}(g) \in I(G)$ gives a continuous homomorphism, whose kernel coincides with the center of $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_k$ be the simple factors of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ (direct sum of ideals). Let $I(\mathfrak{g}), I(\mathfrak{g}_1), \cdots, I(\mathfrak{g}_k)$ be the adjoint groups of $\mathfrak{g}, \mathfrak{g}_1, \cdots, \mathfrak{g}_k$ respectively. $I(\mathfrak{g})$ can be naturally identified with $I(G)$, and we have

$$I(G) = I(\mathfrak{g}) = I(\mathfrak{g}_1) \times I(\mathfrak{g}_2) \times \cdots \times I(\mathfrak{g}_k)$$

(direct product of closed normal subgroups). We denote by $\epsilon$ the identity automorphism of $G$.

**Lemma 1.** Let $G$ be a connected semisimple Lie group. Let $N$ be a nontrivial, i.e., $N \neq \{\epsilon\}$, normal subgroup of $A(G)$. Then there exists an $i$ ($i = 1, 2, \cdots, k$) with $N \ni I(\mathfrak{g}_i)$.

**Proof.** First suppose that $N \cap I(\mathfrak{g}) = \{\epsilon\}$. Let $\sigma$ be in $N$. For $g$ in $G$ we have $\text{Ad}(g) \sigma = \sigma \text{Ad}(g)$, which implies that $\sigma(g^{-1})g$ is in the center of $G$. On the other hand, $G \ni g \mapsto \sigma(g^{-1})g$ gives a continuous map from the connected space $G$. Since the center of $G$ is discrete, we have

Received by the editors September 11, 1967.

1 Research supported in part by NSF Grant GP 4503.
σ(g⁻¹)g = the unit element for all g in G, i.e. σ = e. Hence N = \{e\}. This contradiction implies that N \cap I(\mathfrak{g}) \neq \{e\}.

Since each I(\mathfrak{g}_i) (i = 1, 2, \cdots, k) has no proper normal subgroup, i.e. I(\mathfrak{g}_i) is a simple group, see Goto [3], any nontrivial normal subgroup of I(\mathfrak{g}) = I(\mathfrak{g}_1) \times I(\mathfrak{g}_2) \times \cdots \times I(\mathfrak{g}_k) contains at least one of the I(\mathfrak{g}_i). Q.E.D.

3. Let L be a topological group. For a in L, we denote by C(a) the conjugate class containing a. We define a subset \mathcal{C}(L) of L by the condition: a \in \mathcal{C}(L) if and only if the closure of C(a) is compact. Then the following lemma holds obviously.

**Lemma 2.** \mathcal{C}(L) is a normal subgroup of L.

**Proposition.** Let G be a connected semisimple Lie group without compact components. Then \mathcal{C}(A(G)) = \{e\}.

**Proof.** If it is not true, then by Lemma 1 and Lemma 2 \mathcal{C}(A(G)) must contain some I(\mathfrak{g}_i). Hence it suffices to prove that I(\mathfrak{g}_i) contains a closed subgroup M with \mathcal{C}(M) \neq M.

Since \mathfrak{g}_i is a noncompact simple Lie algebra, there exists a subalgebra \mathfrak{m} of \mathfrak{g}_i, which is isomorphic with sl(2, \mathbb{R}), the Lie algebra of all real 2 by 2 matrices with trace 0. (See Goto [2].) Since I(\mathfrak{g}_i) is a Lie group composed of linear transformations, the analytic subgroup M of I(\mathfrak{g}_i), corresponding to M, is closed and is isomorphic with SL(2, \mathbb{R}), the real special linear group of two dimension, or with I(SL(2, \mathbb{R})). Since the conjugate class containing

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

in SL(2, \mathbb{R}) contains all

\[
\begin{pmatrix}
1 & \alpha \\
0 & 1
\end{pmatrix} 
(\alpha > 0),
\]

we have \mathcal{C}(M) \neq M. Q.E.D.

4. **Proof of Theorem.** Let σ be a continuous automorphism of G. Since σ(h) = h implies that Ad(h)σ = σAd(h), if σ(h) = h for all h in H, then

\[
\{\text{Ad}(g)σ\text{Ad}(g^{-1}); g \in G\} = \{\text{Ad}(k)σ\text{Ad}(k^{-1}); k \in K\}
\]

is compact. Since A(G)/I(G) is finite, C(σ) is compact in A(G). Hence σ \in \mathcal{C}(A(G)), and by the Proposition σ = e. Q.E.D.
In a recent conversation with J. Tits, the author discovered that the main part of this paper is contained in J. Tits, *Automorphismes à déplacement borné des groupes de Lie*, Topology 3 (1964), 97–107.

**Bibliography**


University of Pennsylvania