Introduction. The word "graph" shall mean the graph of a real function, and the $X$-projection of a graph $F$ is the set of all abscissas of points of $F$. $c$ denotes the cardinality of the continuum. If $H$ is a collection of sets, $H^*$ denotes the union of all the sets in $H$.

Definition. Suppose $F$ and $G$ are graphs with $X$-projection $[0, 1]$. The statement that $F$ is dense ($c$-dense) along $G$ means that if $[a, b]$ is a subinterval of $[0, 1]$, then there is a point (are $c$-many points) of intersection of $F$ and $G$ with abscissa in $[a, b]$. In this paper, the following three theorems will be proved:

Theorem 1. If $F$ is a graph with $X$-projection $[0, 1]$, then $F$ is dense along the graph of a function of Baire class 1 with domain $[0, 1]$. However, there is a graph with $X$-projection $[0, 1]$ which is not dense along the graph of any continuous function with domain $[0, 1]$.

Theorem 2. If $F$ is a graph with $X$-projection $[0, 1]$, then $F$ is $c$-dense along the graph of a Lebesgue measurable function with domain $[0, 1]$. However, there is a graph with $X$-projection $[0, 1]$ which is not $c$-dense along the graph of any Baire function with domain $[0, 1]$.

Theorem 3. There exists a graph $F$ with $X$-projection $[0, 1]$ which is $c$-dense along the graph of every Lebesgue measurable function with domain $[0, 1]$.

Proofs. The author is indebted to the referee for the short proof of Theorem 1 which appears here. It makes use of a theorem of H. Blumberg and is considerably shorter than the author's original proof.

Proof of Theorem 1. Suppose $F$ is the graph of a function $f$ with domain $[0, 1]$. It follows from Theorem III of [1] that there is a countable dense subset $D$ of $[0, 1]$ such that $f|D$, the restriction of $f$ to $D$, is continuous. Now, if $f|D$ is bounded, let $g$ be defined as follows: $g(x) = f(x)$ if $x$ is in $D$, and if $x$ is in $[0, 1] - D$, $g(x)$ is the lim sup $f(t)$ as $t \to x$ with $t$ in $D$. Then $g$ is upper semicontinuous and is therefore in Baire class 1, and $F$ is dense along the graph of $g$. Now if $f|D$ is not bounded, let $s_1, s_2, \ldots$ be a sequence of mutually exclusive segments such that $D$ is a subset of $\bigcup s_j$ and $f|D$ is bounded on each $s_j$. For each
positive integer $j$, let $g_j$ be an upper semicontinuous function defined on $s_j$ which agrees with $f$ on $D \cap s_j$ and let $g$ be the function defined as follows: $g(x) = g_j(x)$ if $x$ is in $s_j$ and if $x$ is in $[0, 1] - \cup s_j$, $g(x) = 0$. The function $g$ is in Baire class 1 on the open set $\cup s_j$ and on the closed set $[0, 1] - \cup s_j$. It is therefore in class 1 on all of $[0, 1]$, and $F$ is dense along its graph.

The requirement of the second part of the theorem is satisfied by the graph of any function $f$ with domain $[0, 1]$ such that for some $t$ in $(0, 1)$, $f(t+)$ and $f(t-)$ both exist but are unequal.

Proof of Theorem 2. Suppose $F$ is the graph of a function $f$ with domain $[0, 1]$. Let $M$ be a subset of $[0, 1]$ with measure zero such that if $[a, b]$ is a subinterval of $[0, 1]$, then $[a, b] \cap M$ has cardinality $c$. Let $g(t) = f(t)$ if $t$ is in $M$ and $g(t) = 0$ if $t$ is in $[0, 1] - M$. Clearly, $g$ is Lebesgue measurable, and $F$ is $c$-dense along its graph.

Now, let $R$, the set of all real numbers, be well ordered so that no element of $R$ is preceded by $c$ elements, and let $T$ be a reversible transformation from $[0, 1]$ to the class of all Baire functions with domain $[0, 1]$. For each element $X$ of $[0, 1]$, let $f_X$ denote $T(X)$. Let $h$ be the function defined as follows: for the first element $X$ of $[0, 1]$, $h(X)$ is the first element of $R$ different from $f_X(X)$, and if $Y$ is an element of $[0, 1]$ such that $h(X)$ has been defined for every $X$ in $[0, 1]$ which precedes $Y$, then $h(Y)$ is the first element of $R$ not in $\{f_X(Y) | X \text{ is in } [0, 1] \text{ and } X \text{ precedes } Y \}$. If $F$ is the graph of a Baire function with domain $[0, 1]$ and $H$ is the graph of $h$, then the set of all points of intersection of $F$ and $H$ has cardinality less than $c$. Therefore $H$ is not $c$-dense along the graph of any Baire function with domain $[0, 1]$.

Proof of Theorem 3. Every closed subset of $[0, 1]$ which has positive measure has a perfect subset (the set of its points of condensation) and is therefore of cardinality $c$. The collection of all such sets is of cardinality $c$, so from Theorem 2, p. 458, of [2] it follows that there is a collection $W$ of $c$ mutually exclusive subsets of $[0, 1]$, each of which has $c$ elements in common with each closed subset of $[0, 1]$ of positive measure. Each set of $W$ has outer measure one, or else there would be a closed subset of $[0, 1]$ of positive measure which some set of $W$ would not intersect. Let $T$ be a reversible transformation from $W$ to the class of all Baire functions with domain $[0, 1]$ and for each set $w$ of $W$, let $g_w$ denote $T(w)$. Let $f$ be the function with domain $[0, 1]$ defined as follows: if $t$ belongs to a set $w$ of $W$, $f(t) = g_w(t)$, and if $t$ is in $[0, 1] - W^*$, $f(t) = 0$. Let $F$ be the graph of $f$. Suppose $G$ is the graph of a Lebesgue measurable function $g$ with domain $[0, 1]$ and $[a, b]$ is a subinterval of $[0, 1]$. Let $[d, e]$ be a proper subinterval of $[a, b]$ and $g'$ be a Baire function with domain
[0, 1] which agrees with $g$ except on a set $M$ of measure zero. Let $g''$ be that subset of $g'$ which has domain $[d, e]$. $g''$ is a subset of $c$ different Baire functions $h$ with domain $[0, 1]$, and for each of these a number $t_h$ in $[d, e]$ shall be chosen such that $f(t_h) = g(t_h)$. Let $h$ be a Baire function with domain $[0, 1]$ of which $g''$ is a subset, and let $w$ be $T^{-1}(h)$. Since $w$ has outer measure one, there is a number $t_h$ which is in $w \cap [d, e]$ but not in $M$. Thus $f(t_h) = h(t_h) = g''(t_h) = g'(t_h) = g(t_h)$.

Since the sets in $W$ are mutually exclusive, no two numbers $t_h$ so chosen will be the same. Therefore, the set of points common to $F$ and $G$ which have abscissa in $[d, e]$ and therefore in $[a, b]$ has cardinality $c$.

References


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