A COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS
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1. Statement of results. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic and univalent in the unit disk \( E (|z| < 1) \), then it is known \([1]\) that

\[
|a_3 - \mu a_2^2| \leq 4\mu - 3 \quad \text{when } \mu \geq 1,
\]

\[
\leq 1 + 2 \exp\left[-2\mu/(1 - \mu)\right] \quad \text{when } 0 \leq \mu \leq 1,
\]

\[
\leq 3 - 4\mu \quad \text{when } \mu \leq 0.
\]

The result is sharp in the sense that for each \( \mu \) there is a function in the class under consideration for which equality holds.

This paper contains analogues of (1) for certain classes of analytic functions. Explicitly, let \( \gamma \) and \( \lambda \) be real numbers, where \( |\gamma| < \pi/2 \) and \( 0 \leq \lambda < 1 \), and let \( S(\gamma, \lambda) \) denote the class of analytic functions \( f(z) \) in \( E \) such that \( f(0) = 0 \), \( f'(0) = 1 \) and

\[
\text{Re}\left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} > \lambda \cos \gamma \quad (z \in E).
\]

In particular, \( S(0, \lambda) \) is Robertson’s class of functions that are starlike of order \( \lambda \) in \( E \) \([6]\) and \( S(0, 0) \) is the class of normalized starlike functions. The following sharp result is proved in \( \S 2 \).

Theorem 1. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( S(\gamma, \lambda) \) and if \( \mu \) is a complex number, then

\[
|a_3 - \mu a_2^2| \leq \lambda \cos \gamma \max\left( 1, |2 \cos \gamma(1 - \lambda)(2\mu - 1) - e^{i\gamma}| \right).
\]

For each \( \mu \), there is a function in \( S(\gamma, \lambda) \) for which equality holds.

Hummel ([2], [3]), using variational techniques, proves the conjecture of V. Singh that \( |a_3 - a_2^2| \leq 1/3 \) for the normalized convex functions in \( E \). Since \( zf'(z) \) is starlike if and only if \( f(z) \) is convex in \( E \) \([5, p. 223]\), the following extension of this result is obtained.

Corollary 1. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic and convex in \( E \) and if \( \mu \) is a complex number, then \( |a_3 - \mu a_2^2| \leq \max\left( 1/3, |\mu - 1| \right) \). The result is sharp for each \( \mu \).

Presented to the Society, January 24, 1968; received by the editors July 15, 1967 and, in revised form, October 18, 1967.

1 Supported by the National Science Foundation Grant GP 7377.
A function $f(z)$ is spiral-like [7] in $E$ if there is a real $\gamma$, $|\gamma| < \pi/2$, such that $f(z) \in S(\gamma, 0)$. Another simple consequence of Theorem 1 is

**Corollary 2.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is spiral-like in $E$ and if $\mu$ is a complex number, then

$$|a_3 - \mu a_2^2| \leq 2|\mu - 1| + |2\mu - 1|.$$  

For each real $\mu$, there is a starlike function for which equality holds.

An analytic function $f(z) = z + \cdots$ in $E$ is close-to-convex [4] if there is a real $\gamma$, $|\gamma| < \pi/2$, and a starlike function $g(z) = z + \cdots$ such that

(4) \[ \text{Re} \left\{ e^{i\gamma} \frac{zf'(z)}{g(z)} \right\} > 0 \quad (z \in E). \]

In §3 we prove

**Theorem 2.** If the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $E$ is close-to-convex and if $\mu$ is a real number, then

(5) \[ |a_3 - \mu a_2^2| \leq \max(1, 3|\mu - 1|, 4|\mu - 3|). \]

If $\mu$ is outside the interval $(0, 2/3)$, there is an analytic close-to-convex function for which equality holds.

Let $K_0$ be the subclass of analytic close-to-convex functions $f(z)$ such that (4) holds with $\gamma = 0$ for some starlike function $g(z) = z + \cdots$ in $E$. In §4 we prove the following sharp result.

**Theorem 3.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $K_0$ and if $\mu$ is real, then

(6) \[ |a_3 - \mu a_2^2| \begin{cases} \leq 3 - 4\mu & \text{for } \mu \leq 1/3, \\ \leq 1/3 - 4/9\mu & \text{for } 1/3 \leq \mu \leq 2/3, \\ \leq 1 & \text{for } 2/3 \leq \mu \leq 1, \\ \leq 4\mu - 3 & \text{for } \mu \geq 1. \end{cases} \]

For each $\mu$, there is a function in $K_0$ such that equality holds.

We suspect that the bounds in (6) are sharp when $\mu \in (0, 2/3)$ for the wider class of all analytic close-to-convex functions.

2. **Proof of Theorem 1.** First, if $\Phi(z) = \sum_{n=1}^{\infty} a_n z^n$ is in the class $B$ of functions that are analytic in $E$ and map the unit disk into itself, then $|a_3| \leq 1 - |a_2^2|$ (for example, see [5, p. 108]). Therefore, if $s$ is a complex number, we have
Moreover, the functions \( \Phi(z) = z \) and \( \Phi(z) = z^2 \) respectively show that the result is sharp for \( |s| \geq 1 \) and for \( |s| < 1 \). Now, by (2), \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( S(\alpha, \lambda) \) if and only if the function

\[
\frac{f'(z) - f(z)/z}{f'(z) + [(1 - \lambda)e^{-2i\gamma} - \lambda]f(z)/z} = \sum_{n=1}^{\infty} \alpha_n z^n
\]

is in the class \( B \). A simple computation shows

\[
\alpha_1 = \frac{ua_2}{1 - \lambda}, \quad \alpha_2 = \frac{2u}{1 - \lambda} \left[ \frac{a_3 - 1 - \lambda - u}{2(1 - \lambda)} a_2^2 \right], \quad u = \frac{e^{i\gamma}}{2 \cos \gamma}.
\]

The inequality (3) with

\[
\mu = \frac{1 - \lambda + (s + 1)u}{2(1 - \lambda)}
\]

is now obtained by substituting the coefficients (8) into (7). That (3) is sharp follows from the sharpness of the inequalities (7).

Remark. The same argument also proves

\[
|a_3 - \mu a_2^2| \leq (1 - \lambda) \cos \gamma + (\left| 2 \cos \gamma(1 - \lambda)(2\mu - 1) - e^{i\gamma} \right| - 1) \cdot |a_2|^2 / 4(1 - \lambda) \cos \gamma.
\]

For each \( a_2 \), where \( |a_2| < 2(1 - \lambda) \cos \gamma \), and for each complex number \( \mu \), there is a function in \( S(\gamma, \lambda) \) for which equality holds.

3. Proof of Theorem 2. By (4) the analytic function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in \( E \) is close-to-convex if and only if there exists \( a g(z) = z + \sum_{n=2}^{\infty} c_n z^n \) in \( S(0, 0) \) such that the function

\[
\Phi(z) = e^{i\gamma} \frac{f'(z) - g(z)/z}{f'(z) + e^{-2i\gamma} g(z)/z} = \sum_{n=1}^{\infty} \alpha_n z^n
\]

is in the class \( B \) of \( \S 2 \). A comparison of the coefficients in the various power series expansions for the functions in this identity shows

\[
2a_2 = c_2 + 2 \cos \gamma a_1, \quad 3a_3 = c_3 + 2 \cos \gamma(\alpha_1 c_2 + a_2 + e^{i\gamma} a_1).\]

Therefore, we have

\[
\alpha_3 - \mu a_2^2 = \frac{1}{3}(c_3 - \frac{2}{3} \mu c_2) + \frac{2}{3} \cos \gamma[a_2 + (e^{i\gamma} - \frac{3}{2} \mu \cos \gamma)\alpha_1] + \frac{1}{3}(\mu - \frac{3}{2}) \cos \gamma a_1 c_2.
\]
Set \( \mu = 2/3 \). By (7) and Theorem 1, we obtain
\[
| a_3 - \frac{2}{3} \alpha_2 | \leq \frac{1}{3} \left| \alpha_3 - \frac{1}{3} \alpha_2 \right| + \frac{2}{3} \cos \gamma \left| \alpha_2 + i \sin \gamma \alpha_1 \right| \\
\leq \frac{1}{3} + \frac{2}{3} \cos \gamma \leq 1.
\]
From the Area Theorem \( [5, p. 210] \), we have \(| a_3 - a_2 | \leq 1 \) and by (9), we get \(| a_3 | \leq 3 \). Thus for \( 0 \leq \mu \leq 2/3 \), it follows that
\[
| a_3 - \mu a_2 | \leq \frac{2}{3} \mu \left| a_3 - \frac{2}{3} a_2 \right| + (1 - \frac{2}{3} \mu) | a_3 | \leq 3(1 - \mu)
\]
and, for \( 2/3 \leq \mu \leq 1 \), that
\[
| a_3 - \mu a_2 | \leq (3 \mu - 2) | a_3 - a_2 | + 3(1 - \mu) | a_3 - 2a_2/3 | \leq 1.
\]
The last result is sharp since the close-to-convex class include the starlike functions \( S(0, 0) \) and the inequality is sharp in the latter class by Theorem 1. Finally, if \( \mu \) is not in the interval \( [0, 1] \), then by (1) \(| a_3 - \mu a_2 \| \leq |4 \mu - 3| \) since the close-to-convex functions are univalent \([4]\).

4. **Proof of Theorem 3.** From (7), (9) with \( \gamma = 0 \) and Theorem 1 for the starlike class, we have
\[
| a_3 - \mu a_2 | \leq \frac{1}{3} \left\{ 1 + \frac{1}{12} \left[ | 3 \mu - 3 | - 1 \right] \left| \alpha_2 \right| \right\} \\
+ \frac{5}{3} \{ 1 + \frac{1}{3} \left| \alpha_1 \right| | c_1 | \}
\]
\[
+ \frac{5}{3} | 3 \mu - 2 | \left| \alpha_1 \right| | c_1 |.
\]
If \( 1/3 \leq \mu \leq 2/3 \), this becomes
\[
| a_3 - \mu a_2 | \leq 1 + \frac{1}{12} \{ (2 - 3 \mu) \left| \alpha_2 \right| \\
+ 4(2 - 3 \mu) \left| \alpha_1 \right| \left| c_1 \right| - 12 \mu \left| \alpha_1 \right| \}
\]
\[
= 1 + \frac{1}{12} \left\{ 2 - 3 \mu + \frac{(2 - 3 \mu)^2}{3 \mu} \right\} \left| \alpha_2 \right| \\
- \mu \left\{ \left| \alpha_1 \right| - \frac{(2 - 3 \mu)}{6 \mu} \left| c_2 \right| \right\}^2
\]
\[
\leq 1 + \frac{2 - 3 \mu}{18 \mu} \left| c_2 \right| \leq \frac{1}{3} + \frac{4}{9 \mu},
\]
since \( \left| c_2 \right| \leq 2 \). The result is sharp since there is a starlike function (the Koebe function \( g(z) = z/(1 - z)^2 \)) with \( c_2 = 2, c_3 = 3 \) and a function in \( B \) with \( \alpha_1 = (2 - 3 \mu)/3 \mu, \alpha_2 = 1 - \alpha_1^2 \), provided \( 1/3 \leq \mu \leq 2/3 \). For \( 0 \leq \mu \leq 1/3 \), we have
\[ |a_3 - \mu a_2^2| \leq 3\mu |a_3 - a_2^2/3| + (1 - 3\mu) |a_3| \leq 3 - 4\mu. \]

For the remaining choices of \(\mu\), (6) is a consequence of Theorem 2. The sharpness for \(\mu\) not in the interval \((1/3, 2/3)\) follows from Theorem 1, since \(S(0, 0) \subset K_0\).

**References**


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