A UNIQUENESS THEOREM FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS

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In an earlier paper, it was shown, using a modified energy integral technique, that the Dirichlet and Neumann boundary value problems for linear differential equations of a certain class always have a unique solution [1]. In this paper, a theorem is proved which extends the linear uniqueness theorem to some nonlinear boundary value problems.

In the following the domains of the variable \( x = (x_1, x_2, \ldots, x_n) \) and the parameter \( z \) are taken to be \( X \) and \( D \) respectively. Their boundaries are written \( \partial X \) and \( \partial D \). A subdomain of \( D \), either proper or improper, is denoted by \( D^* \). The set \( U_F(\lambda, \mu) \) is defined to be the collection of pairs of complex numbers \((\lambda_a, \mu_a)\) which are such that

(i) \( F(x, z, \lambda_a, \mu_a) \) is analytic in \( z \) on \( D \times X \)

(ii) there exist two real valued functions \( p_\lambda(x, z) \) and \( p_\mu(x, z) \) defined on \( X \times \partial D \) such that

\[
| F(x, z, \lambda_a, \mu_a) - F(x, z, \lambda, \mu) | < p_\lambda(x, z) | \lambda_a - \lambda | + p_\mu(x, z) | \mu_a - \mu | .
\]

In the following, the function \( v(x, z_0) \) \((z_0 \in D^*)\) will be said to be \( F^{(w)} \)-admissible if \((v, \nabla_x v) \in U_F(u, \nabla u)\) and both \( v \) and \( \nabla_x v \) are square integrable \((\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n))\). In the complex number plane, half planes whose boundaries contain the origin shall be denoted by \( H \). The outward drawn unit normal to such a half plane is \( v_H \).

The operator

\[
L = - \nabla_x \cdot a(x, z) \nabla_x + b(x, z)
\]

with \( a(x, z) \) and \( b(x, z) \) analytic in \( z \) on \( D \times X \) is of class \( H_0 \) if and only if

(i) for each \( z \in \partial D \) and every pair of real numbers \((\xi, \eta) \neq (0, 0)\), there exists an \( H \) such that the mapping \( h_{ab}(\xi, \eta) \) of \( X \)

\[
h_{ab}(\xi, \eta) = a(x, z) \xi^2 + b(x, z) \eta^2
\]

is contained in the interior of \( H \), and

Received by the editors September 14, 1967.

\(^1\) This work was supported in part by National Science Foundation Contract GP 7543.
(ii) the winding number (with respect to the origin of the complex plane) of $v_H$, defined as a continuous function of $z$ on $\partial D$, is zero.

Consider the equation

$$L[u] = F(x, z, u, \nabla_z u).$$

Problem $N$ will be to solve equation (4) on $D^* \times X$ together with a Neumann condition on $D^* \times \partial X$. Problem $D$ will be to solve equation (4) on $D^* \times X$ together with a Dirichlet condition on $D^* \times \partial X$.

**Theorem.** Assume that $L \in H_0$. Assume that on $X \times \partial D$ there exists a real function $\theta(z)$, which is such that

$$\frac{1}{2} p_\mu(x, z) \leq \min_{x \in X} \text{Re}\{a(x, z)e^{-i\theta(z)}\},$$

$$\frac{1}{2} p_\mu(x, z) + p_\lambda(x, z) \leq \min_{x \in X} \text{Re}\{b(x, z)e^{-i\theta(z)}\}.$$

Then any $F^{(u)}$-admissible function $v(x, z)$ which solves either problem $N$ or problem $D$ is unique in $U_{F(u, \nabla u)}$ if $u$ is a solution.

**Proof.** Assume that an $F^{(u)}$-admissible function $u$ solves problem $N$ (or $D$) and that there exists a second $F^{(w)}$-admissible solution $u_0$ for some value of $z_0 \in D^*$. Then their difference $\psi(x, z_0) = u_0(x, z_0) - u(x, z_0)$ solves the equation

$$L(\psi) = F(x, z, u_0, \nabla_z u_0) - F(x, z, u, \nabla_z u) = \mathcal{F}(x, z, u_0, u)$$

when $z = z_0$, together with appropriate data equal to zero on $D^* \times \partial X$. Therefore, multiplying equation (6) by $\overline{\mathcal{F}(x, z_0)}$, the complex conjugate of $\mathcal{F}(x, z_0)$, and integrating by parts yields

$$0 = \int_X dx \left[ h_{ab}(\psi, \nabla_z \psi) \right] - \int_X dx \mathcal{F}(x, z, u_0, u).$$

Define the function

$$S(z, z_0) = \int_X dx \left[ a(x, z) \nabla_z \psi(x, z_0) \right]^2 + b(x, z) \psi(x, z_0)^{\frac{1}{2}}$$

$$- \int_X dx \left[ \mathcal{F}(x, z_0) \mathcal{F}(x, z, u_0(x, z_0), u(x, z_0)) \right].$$

Since $u_0$ and $u$ are admissible, $S(z, z_0)$ is analytic in $z$ on $D$. From equation (6) we conclude that $S(z_0, z_0) = 0$. From equation (1), the absolute value of the second integral is less than

$$\int_X dx \left[ \left\{ p_\lambda(x, z) + \frac{1}{2} p_\mu(x, z) \right\} \right] \psi(x, z_0)^{\frac{1}{2}}.$$
which is less than the absolute value of the first integral on $X \times \partial D$. By Rouché's theorem, the number of zeros of $S(z, z_0)$ and the number of zeros of the first integral in $D$ are the same. However, it has already been shown in an earlier paper that if $L \in H_0$, the first integral cannot vanish for any value of $z \in D$ when $\psi(x, z)$ is the difference of two admissible functions, unless $\psi(x, z_0) = 0$ almost everywhere [1]. Thus, the solution is unique.

**Reference**


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