A NOTE ON THE LITTLEWOOD-TAUBER THEOREM

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The application of Wiener's generalized Tauberian Theorem [1] to a sequence of functions in \( L_1(-\infty, \infty) \) leads to the following.

**Lemma.** Let \( K_0 \) belong to \( L_1(-\infty, \infty) \) and be such that

\[
\int_{-\infty}^{\infty} K_0(x) e^{iux} dx \neq 0, \quad -\infty < u < \infty.
\]

Let \( g \) be bounded on \((-\infty, \infty)\) and let

\[
\lim_{z \to \infty} \int_{-\infty}^{\infty} K_0(x - y) g(y) dy = A \int_{-\infty}^{\infty} K_0(y) dy.
\]

Then if \( \{K_n(x)\} \) is a sequence of functions in \( L_1 \) such that for \( n \to \infty \)

\[
\int_{-\infty}^{\infty} K_n(x - y) g(y) dy \to I(x)
\]

for almost all \( x \) and

\[
\int_{-\infty}^{\infty} K_n(y) dy \to I,
\]

we have

\[
\lim_{z \to \infty} I(x) = AI.
\]

As an application of this lemma we give a particularly simple proof of the Littlewood-Tauber Theorem [2].

**Theorem.** Let \( \sum_{0}^{\infty} a_n x^n \) converge to \( f(x) \) for \( |x| < 1 \) and let

\[
\lim_{z \to 1-} f(x) = s < \infty.
\]

If \( n |a_n| < K < \infty \), then \( \sum_{0}^{\infty} a_n = s \).

**Proof.**

\[
\left| \sum_{0}^{N} a_n - \sum_{0}^{\infty} a_n e^{-n/N} \right| \leq \sum_{0}^{N} |a_n| (1 - e^{-N}) + \sum_{N+1}^{\infty} |a_n| e^{-n/N} \\
\leq K \left( 2 + \sum_{N+1}^{\infty} \frac{e^{-n/N}}{n} \right) < \infty \quad \text{for } N > 1,
\]

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hence \( s(u) = \sum_{0}^{[u]} a_n \) is bounded. Now,

\[
f(e^{-z}) = \int_{0}^{\infty} e^{-uz} s(u) du
\]

\[
= \int_{0}^{\infty} e^{-(t-\nu)} e^{-e^{-(t-\nu)} s(\nu)} dy,
\]

where \( x = e^{-z} \). Thus,

\[
s = \lim_{x \to 0^+} \int_{\xi}^{\infty} K_0(\xi - y) s(\nu) dy,
\]

where \( K_0(x) = e^{-x} e^{-e^{-x}} \). We have

\[
\int_{-\infty}^{\infty} K_0(x) e^{iux} dx = \Gamma(1 - iu) \neq 0.
\]

Let \( K_n(x) = (n/\pi)^{1/2} e^{-nx^2} \), then the conditions of the Lemma are satisfied under the hypothesis that \( n | a_n | \) is bounded and, therefore, noting that in this case \( I(x) = g(x) \), \( I = 1 \), we have

\[
s = \lim_{x \to \infty} \lim_{n \to \infty} \int_{-\infty}^{\infty} K_n(x - y) s(\nu) dy = \lim_{x \to \infty} s(e^z) = \sum_{0}^{\infty} a_n.
\]

**References**

1. N. Wiener, *The Fourier integral*, Dover, New York, 1933; p. 73.

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