A THEOREM ON INFINITE POSITIVE MATRICES

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1. Let $A = (a_{ij})$ be an infinite matrix with positive elements $a_{ij} > 0, i, j = 0, 1, \ldots$, (matrices $(a_{ij})$ with $a_{ij} > 0$ will be called in the sequel positive matrices).

It was proved in [3], that

(1) if $A$ is a finite positive matrix, a unique doubly stochastic matrix $T$ exists such that $T = D_1 A D_2$ where $D_1$ and $D_2$ are diagonal matrices with all elements on the diagonal positive and are unique up to a scalar factor.

The method used in [3], and introduced first in [4], is a constructive one and consists in alternate normalizing rows and columns of $A$ and proving the convergence of this procedure. Another proof of (1) was given in [1]. This second proof uses besides Brouwer’s fixed point theorem the fact, that

(2) the set \( \{ x = (x_0, x_1, \cdots, x_n); x_i \text{ real numbers, } \sum_{i=0}^{n} x_i^2 = 1 \text{ and } x_i \geq 0 \} \) is homeomorphic to an $n$-dimensional ball.

Although a purely existential one, this second proof contains a statement about the existence of directions of fixed points for some mapping defined by help of a finite matrix $A$. In this paper we note that statement (1) does not hold for infinite matrices and prove a theorem generalizing properly (1) to the case of infinite matrices. Essentially, both proofs in [1] and in [3] could be, with some non-trivial changes, applied to give the desired generalization. The difficulty in generalizing the proof given in [3] consists i.a. in the fact that for an infinite matrix $\sum_j a_{ij}$ (or $\sum_i a_{ij}$) is not always finite. The idea of our proof is similar to that of [1] except that (2) is not used and that Brouwer’s theorem is replaced by the theorem of Schauder (see [2]). In the sequel a matrix $A = (a_{ij})$ with $a_{ij} > 0$ will be called a positive matrix and a diagonal matrix with positive diagonal elements will be called a positive diagonal matrix. Finally $\delta_{ij} = 0, i \neq j; 1, i = j$, will denote the delta of Kronecker.

2. Before generalizing (1) to the case of infinite matrices let us note that

(3) if $A = (a_{ij})$ is infinite with $a_{ij} = 1$ for $i, j = 0, 1, 2, \cdots$ then positive diagonal matrices $D_1$ and $D_2$ for which $T = D_1 A D_2$ is doubly stochastic do not exist.

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Indeed, if $D_1 = (\delta_{ij} p_i)$ and $D_2 = (\delta_{ij} q_j)$, then for a doubly stochastic matrix $T = D_1 A D_2$ one has $t_{ij} = p_i q_j$ and $p_i = p_j$ for all $i, j = 0, 1, \cdots$, which is impossible.

We show now, that

(4) if $A = (a_{ij})$ with $a_{ij} = 1$ for all $i, j = 0, 1, \cdots$ and if there exist positive diagonal matrices $D_1 = (\delta_{ij} \alpha_i)$ and $D_2 = (\delta_{ij} \beta_j)$ such that for $T = D_1 A D_2 = (t_{ij})$ one has $\sum_j t_{ij} = \alpha_i$ and $\sum_i t_{ij} = \beta_j$, $i, j = 0, 1, \cdots$, then $\sum_{i=0}^{\infty} \alpha_i = \sum_{j=0}^{\infty} \beta_j < \infty$.

Indeed, condition $\sum_{j=0}^{\infty} t_{ij} = \alpha_0$ i.e. $p_0 \sum_{j=0}^{\infty} q_j = \alpha_0$ implies $\sum_{j=0}^{\infty} q_j < \infty$ and similarly $\sum_{i=0}^{\infty} p_i < \infty$. But then $(\sum_{i=0}^{\infty} p_i)(\sum_{j=0}^{\infty} q_j) = \sum_{i=0}^{\infty} \alpha_i = \sum_{j=0}^{\infty} \beta_j < \infty$.

Property (4) justifies the assumption $\sum_{i=0}^{\infty} \alpha_i = \sum_{j=0}^{\infty} \beta_j < \infty$ made in the following

**Theorem.** Let $A = (a_{ij})_{i,j=0,1,\ldots}$ be an infinite positive matrix such that

(a) there exists a constant $M$ with $a_{ij} \leq M$ and

(b) there exists a column (say the 0th column) and constants $L_0$ and $M_0$ such that for every $i, k = 0, 1, \cdots$ one has $a_{i0} \leq M_0 a_{ik}$ and $a_{0k} \leq L_0 a_{i0}$.

Let further $\{\alpha_i\}$ and $\{\beta_j\}$ be sequences of positive numbers such that $\sum_{i=0}^{\infty} \alpha_i = \sum_{j=0}^{\infty} \beta_j < \infty$.

Then there exist positive diagonal matrices $D_1$ and $D_2$ such that for $T = D_1 A D_2 = (t_{ij})$ one has

(c) $\sum_i t_{ij} = \alpha_i$ and $\sum_j t_{ij} = \beta_j$, $i, j = 0, 1, \cdots$.

**Proof.** Putting $N_0 = 1$ and $N_i = N \geq 1$ for $i \geq 1$ and multiplying $A$ on the right by the matrix $D = (\delta_{ij} N_i)$ one can by choosing $N$ sufficiently large obtain by (b) that, for the matrix $B = AD = (b_{ij})$,

(d) $b_{i0} \leq b_{ik}$ and $b_{0k} \leq (\beta_0/\sum_{k=1}^{\infty} \beta_k) b_{0k}$ holds for every $i, k = 0, 1, \cdots, k \neq 0$.

It suffices obviously to find positive diagonal matrices $P$ and $Q$ such that $T = PBQ = (t_{ij})$ satisfies (c) (then $D_1 = P$ and $D_2 = DQ$). Now consider the equations

(e1) $u_i \sum_j b_{ij} x_j = \alpha_i$,

(e2) $x_k \sum_i b_{ik} u_i = \beta_k$, $i, k = 0, 1, \cdots$.

Expressing $x_k$ in terms of $x_j$ we get

(f) $x_k = \beta_k/\sum_{x \neq k} f_k \{x_j\}$, where $f_k \{x_j\} = \sum_{x=0}^{\infty} (\alpha_i b_{ik}/\sum_{j=0}^{\infty} b_{ij} x_j)$.

Evidently, if one finds a sequence $\{x_j\}$ with $x_j > 0$ such that in (f) $x_k = x_k'$ for every $k \geq 0$, then calculating $u_i$ from (e1) and putting $P = (\delta_{ij} x_i)$ and $Q = (\delta_{ij} x_j)$ we have the desired matrices $P$ and $Q$. In other words one looks for any fixed point $x = \{x_k\}_{k=0,1,\ldots}$, $x_k > 0$ of the mapping defined by (f). To get such a fixed point let us denote
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\[ \xi_k = x_k / x_0 \] and \( \eta_k = x_k / x_0 \), \( k = 1, 2, \ldots \), (we call \( \{\xi_k\} \) and \( \{\eta_k\} \) “directions”).

Then by (f) one has

\[ \eta_k = \frac{\beta_k}{\beta_0} \frac{g_0(\xi_j)}{g_k(\xi_j)}, \quad \text{where} \quad g_k(\xi_j) = \sum_{i=0}^{\infty} \alpha_i \delta_{ik} b_{i0} + \sum_{j=1}^{\infty} b_{ij} \xi_j. \]

Let us confine ourselves to \( \xi_j \geq 0 \) such that \( \sum_{j=1}^{\infty} \xi_j \leq 1 \), i.e. such that the point \( x = (\xi_1, \xi_2, \ldots) \) belongs to the intersection \( C \cap S \) of the cone \( C = \{x = (\xi_1, \xi_2, \ldots); \xi_i \geq 0, x \in l\} \) in the Banach space \( l \) with the unit ball \( S \) of this space.\(^2\) Since \( \sum \alpha_i < \infty \) we obtain by (a) that \( \eta_k \) exists for all \( k = 1, 2, \ldots \) and obviously by (b) \( \eta_k > 0 \). By (d) it follows that \( \eta_k \leq \beta_k / \beta_0 \) and that \( \sum_{k=1}^{\infty} \eta_k \leq 1 \).

Thus, by \( \sum \beta_k < \infty \), formula (g) defines a continuous mapping \( F \) of \( C \cap S \) into a compact subset of \( C \cap S \). By the fixed point theorem of Schauder (see [2]) there exists a point \( \bar{x} = (\xi_1, \xi_2, \ldots) \) such that \( F(\bar{x}) = \bar{x} \). This point is an invariant direction of the mapping \( F \). Take as in [1] any point \( (x'_1, x'_2, \ldots) \) on this direction with \( x'_j > 0 \). Then \( x_j = \theta x'_j \), \( j = 0, 1, \ldots \), and putting \( x'_j \) into (e) we obtain the sequence \( \{u_i\} \). Then by (e1) and (e2) we have

\[ \sum_i u_i \sum_j b_{ij} x'_j = \sum \alpha_i = \sum \beta_i = \theta \sum u_i \sum_j b_{ij} x'_j. \]

Thus \( \theta = 1 \) and the sequences \( \{u_i\} \) and \( \{x_j\} \) satisfy both (e1) and (e2).

The theorem is proved.

REMARKS. Let us note that in case of a finite positive matrix \( A = (a_{ij}) \) all the assumptions of the Theorem hold. Finally let us note that if \( A = (a_{ij}) \) is infinite and \( a_{ij} = 1/2^{i+j} \), \( i, j = 0, 1, 2, \ldots \), then \( A \) is obviously a stochastic matrix but the argument applied in (3) shows that positive diagonal matrices \( D_1 \) and \( D_2 \) for which \( T = D_1 A D_2 \) is doubly stochastic do not exist.

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\(^2\) \( l \) denotes the Banach space of all sequences \( x = (\xi_1, \xi_2, \ldots) \) with \( \xi_i \) real and \( \sum |\xi_i| < \infty \).