RIGIDITY OF GENERALIZED UNISERIAL AND FROBENIUS ALGEBRAS

S. S. PAGE

Introduction. In [2], [4] Gerstenhaber, Nijenhuis and Richardson introduced the concept of deformations of an algebra $A$ over a field. In this note we consider the question of whether there are any reasonable classes of algebras which admit nontrivial deformations but for which all the deformations remain in the class. The main theorem states that if $A$ is a generalized uniserial basic algebra, then every deformation of $A$ is generalized uniserial. The theorem actually gives a complete description of all deformations of a generalized uniserial basic algebra.

The second theorem states that the class of Frobenius algebras is a class closed under deformations.

We begin by setting up the notation. Throughout $A$ will denote an associative algebra over a field $k$ which admits a Wedderburn decomposition, $A = S + N$ where $S$ is $K$ separable and $N$ is the Jacobson radical. $A$ will be called a basic algebra if the simple components of $A$ are one dimensional over $k$ and in this case we will write $S = \sum_{i=1}^{n} ke_i$ where $e_i$ is the identity of the $i$th component of $S$.

A generalized uniserial algebra $A$ is an algebra such that for any primitive idempotent $e$ the left (resp. right) modules $Ae$ ($eA$) have a unique decomposition series.

Following [2], $A_0 = A \otimes_k k((t))$ where $k((t))$ is the field of quotients of the power series ring over $k$ in one indeterminant $t$. By a deformation of $A$ we will mean an associative multiplication induced by a bilinear function $f_t: A \otimes_k A \rightarrow A_0$ of the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) \cdots .$$

Call such functions $f_t$ multiplicative functions. Two multiplicative functions $f_t$ and $g_t$ are said to be equivalent, $f_t \sim g_t$, if there exists a linear function $\psi_t: A \rightarrow A_0$ of the form $\psi_t(a) = a + \varphi_1(a)t + \varphi_2(a)t^2 + \cdots$ such that $f_t(a, b) = \psi_t^{-1}(g_t(\psi_t(a), \psi_t(b)))$. It is known, see [5], [7], that every deformation given by a multiplicative function $f_t = ab + tF_1(a, b) \cdots$ is equivalent to a deformation given by a multiplicative function $g_t(a, b) = ab + tG_1(a, b) + t^2G_2(a, b) \cdots$ where $G_t(a, S) = G_t(S, a) = 0$ for all $s \in S$ and $a \in A$.

The first theorem we need is due to Kupisch [3].

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**Theorem.** Let $A$ be a generalized uniserial basic algebra. Then $A = S + M^{(1)} + M^{(2)} + \cdots + M^{(n)}$ (K-direct) where $M^{(1)} + N^2 = N$ and $M^{(n)} + N^{n+1} = N^n$ and $A$ has a basis $\{b_i\}_{i=1}^n$ such that $b_1 = e_1$, $\cdots$, $b_{p_0} = e_{p_0}$, $b_{p_0+1} \in N^{(1)}$, $\cdots$, $b_{p_1} \in N^{(1)}$, $b_{p_1+1} \cdots b_{p_2} \in N^{(2)}$, $\cdots$, $b_{p_n} \in N^{(n)}$ and the nonzero products of the $b_{p_{i+1}} \cdots b_{p_i}$, with the $b_{p_{i+1}} \cdots b_{p_i}$ are the $b_{p_{i+1}} \cdots b_{p_i}$ and for each $b_a$ there exists a unique $i$ such that $e_i b_a = b_a$ and a unique $j$ such that $b_a e_j = b_a$. We will write $e_i m e_j$ for the $b_{p_{i+1}} \cdots b_{p_i}$.

We will need also the following lemma the proof of which is routine and will be omitted.

**Lemma.** Let $K$ be any field and let $A = K[x]/I$, $I$ any ideal of $K[x]$. Then $A$ is generalized uniserial.

We now prove two lemmas the first of which is a special case of the desired result and the second a reduction theorem.

**Lemma.** The only deformations of $L[x]/(x^n)$ are of the form $L[x]/P(x)$ where $P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$ and $L = K((t))$.

**Proof.** Since $\phi \in H^2(A, A)$ we claim $\phi = 0$, and letting $G_1 = F_1 - \delta\phi$.

Notice now if we set $g_i(a, b) = ab + t G_1(a, b)$ we have

$$A_{g_i} \cong L[x]/(x^n + t G_1(x, x^{n-1})).$$

Thus $G_1(a, G_1(b, c)) - G_1(G_1(a, b), c) = 0$, $a, b, c \in K[x]/(x^n)$.

Now let $f_i$ be a multiplicative function of the form

$$f_i(a, b) = ab + t F_1(a, b) + t^2 F_2(a, b) \cdots$$

then by the above $f_i \sim f_i'$ where

$$f_i'(a, b) = ab + t G_1(a, b) + t^2 G_2(a, b) \cdots$$

and $G_1(x, x^j) = 0$, $j < n - 1$, and $G(a, G(b, c)) - G(G(a, b), c) = 0$ for all $a, b, c$, in $K[x]/(x^n)$. By this last equation we see that $\delta G_2 = 0$, so we can repeat the process to $G_2$, so that $f_i \sim f_i''$ where

$$f_i''(a, b) = ab + t G_1(a, b) + t^3 G_2(a, b) + \cdots$$

where $G_2(x, x^j) = 0$, $j < n - 1$; and it follows that $G_2 = 0$. Continuing by induction we find $f_i \sim h_i$ where

$$h_i(a, b) = ab + t H_1(a, b) + t^2 H_2(a, b) \cdots$$
and \( H_i(x, x^j) = 0, j < n - 1 \). Now set

\[ p(x) = x^n + tH_1(x, x^{n-1}) + t^2H_2(x, x^{n-1}) \ldots. \]

One can easily verify that \( A_n \cong \mathbb{L}[x]/(p(x)) \).

**Lemma.** Let \( A \) be a generalized uniserial basic algebra where \( N = M + M^2 + M^3 + \cdots + M^n \) as in Theorem 3.9. Then if \( F \in H^2(A, A) \) and \( F(M^i, M^j) \in \sum_{i=1}^{n} M^i \), \( F \) is trivial.

**Proof.** Let \( e_i m_j = i m_j \) be the basis of \( M \) from Kupisch's Theorem. On the basis of \( M^2 \) define

\[ \phi_1(m) = -F(e_i m_j, i m_k), \quad m = i m_j \text{ or } i m_k, \]

\[ = 0 \quad \text{otherwise} \]

and extend \( \phi_1 \) linearly to all of \( A \) in the obvious way. Now \( G_1 = F + \delta \phi_1 \) has the property that \( G_1(M, M) = 0 \). Define \( \phi_2 \) on the basis of \( M^3 \) by

\[ \phi_2(m) = -G(e_i m_j, m_k), \quad m = i m_j \text{ or } i m_k, \]

\[ = 0 \quad \text{otherwise} \]

and extend to all of \( A \).

Setting \( G_2 = G_1 + \delta \phi_2 \), \( G_2 \) has the property that \( G_2(M^2, M) = 0 \). But since \( \delta G_2 = 0 \), \( G_2(M, M^2) = 0 \). Continuing inductively we obtain \( F \sim G \) where \( G(M, N) = 0 \). It follows that \( G = 0 \) since \( \delta G = 0 \).

**Theorem.** If \( A \) is a generalized uniserial basic algebra, then all deformations of \( A_0 \) are generalized uniserial.

**Proof.** Let \( F_i \) be a multiplicative function. We can assume that \( f_i \) has the form \( f_i(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) \) \ldots and \( F_1 \) is not zero in \( H^2(A, A) \). Let \( F = F_1 + F_2 \) where

\[ F_1(M^i, M^j) \in \sum_{k \ge i+j} M^k, \quad F_2(M^i, M^j) \in \sum_{k \ge i+j+1} M^k. \]

One checks easily that \( F_1 \) and \( F_2 \in H^2(A, A) \), and therefore we can assume \( F_2 = 0 \) by the above.

Now since \( F \) is nontrivial it follows that \( F(M, N) \neq 0 \); so there exists \( m_j \) such that \( F(m_j, M^i) \neq 0 \). Pick \( m_j \) so \( l \) is minimal with this property. Now for some \( m_k \cdots m_y \in M \) we have

\[ F(m_j, m_k m \cdots m_y) = m_j m_y \neq 0, \]

where \( y_y \in \sum_{i \le k \le y-1} M^k \). Set
\[ \phi(z) = jy_p \quad z = \lambda m_k m \cdots \lambda m_p, \]
\[ = 0 \quad \text{otherwise,} \]

extending to all of \( A \). Form \( F - \delta \phi = G \). \( G \) has the property that \( G(jm_k, km \cdots \lambda m_p) \neq 0 \) and note that \( \lambda m_q \cdots \lambda m_p \subseteq M^{l-1} \). Repeating the process to \( G \) we find \( F \sim G_1 \) where \( G_1(m_j, \lambda m_p) \neq 0 \). But then

\[ G_1(m_j, \lambda m_p) = \alpha \lambda m_p \quad \text{some } \alpha \in K, \]

but by the uniqueness of the idempotents on the left we have \( j = i \), which implies \( j = p \), which implies \( i = p \). So in the above we really had

\[ G_1(p m_p, \lambda m_p) = \alpha \lambda m_p. \]

This implies that \( A e_p \) is a ring direct summand of \( A \) and each \( F_i(Ae_p, Ae_p) \subseteq Ae_p \). Now \( Ae_p = e_pAe_p \cong K[x]/(x^n) \) some \( n \); so it follows that we are only deforming ring direct summands which are of the form \( K[x]/(x^n) \), and they remain ring direct summands. The result follows.

We now drop all assumptions on \( A \) except that \( A \) be finite dimensional over \( K \).

**Theorem.** Let \( A \) be a Frobenius algebra over \( k \). Then every deformation of \( A \) is Frobenius as an algebra over \( K((t)) \).

**Proof.** Since \( A \) is Frobenius there exists a linear functional \( \lambda: A \to K \) such that there are no ideals (right or left) in \( \ker \lambda \) [1, p. 414]. \( A \otimes k((t)) \) is Frobenius and in fact if we define \( \lambda(a \otimes \lambda(t)) = \lambda(a) \lambda(t) \) kernel \( \lambda \) contains no ideals in \( A \otimes k((t)) \). Now let \( f_i \) be a deformation of \( A \). \( \lambda \) is still a linear functional on the deformed algebra and one easily checks that kernel \( \lambda \) contains no ideals. Thus \( A_{f_i} \) is Frobenius.

For further theorems on properties perserved under deformations see [5], [6].

**Bibliography**


University of British Columbia