WEAK COMPACTNESS OF MEASURES

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1. Introduction. This paper is concerned with a description of the weakly relatively compact subsets of the space of regular Borel measures on a compact Hausdorff space $X$. Several characterizations of such sets are known through the work of Pettis [6], Grothendieck [4], and Dieudonné [2]. We find a weak set of Boolean conditions on a family of open sets of $X$ to insure that convergence of a sequence of measures on each member of the family implies weak convergence of the sequence. This result is then applied to the Boolean algebra of regular open sets of $X$ to obtain a generalization to arbitrary compact Hausdorff spaces of a theorem of Grothendieck on Stonian spaces.

W. G. Bade has remarked that Grothendieck's theorem is equivalent to a well-known lemma of R. S. Phillips concerning the equivalence of weak convergence and weak-star convergence in $l^*_e$. Thus our generalization provides a new proof of Phillips' Lemma.

2. Preliminaries. Let $X$ be a compact Hausdorff space. Denote the Banach space of all real or complex-valued continuous functions on $X$ by $C(X)$, and denote the Banach space of all regular Borel measures on $X$ by $M(X)$. The dual space of $C(X)$ is $M(X)$, and if $\mu \in M(X)$ then $||\mu|| = |\mu|(X)$, the total variation of $\mu$ on $X$. The topology for $M(X)$ of pointwise convergence on $C(X)$ is called the weak star topology and is denoted by $\sigma(M(X), C(X))$. The topology for $M(X)$ of pointwise convergence on $M(X)^*$, the dual of $M(X)$, is called the weak topology and is denoted by $\sigma(M(X), M(X)^*)$. From the Eberlein-Smulian Theorem we know that a subset $K$ of $M(X)$ is weakly relatively compact iff every sequence in $K$ has a weakly converging subsequence. Also useful is the fact that $M(X)$ is a weakly complete space. The classical necessary and sufficient condition for weak convergence of a bounded sequence $\{\mu_n\}$ in $M(X)$ is that $\lim_n \mu_n(E)$ exists for each Borel set $E \subseteq X$, [3, p. 308]. Two basic results in this connection are

Theorem 1 (Grothendieck [4, p. 147]). A sequence $\{\mu_n\}$ in $M(X)$ is $\sigma(M(X), M(X)^*)$-convergent iff for every sequence $\{E_n\}$ of pairwise disjoint open sets of $X$ $\lim_n \mu_n(E_n) = 0$ uniformly in $n$. 

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Theorem 2 (Dieudonné-Grothendieck). A sequence \( \{\mu_n\} \) in \( M(X) \) is convergent for the \( \sigma(M(X), M(X)^*) \) topology iff for each open set \( G \subseteq X \), \( \lim_n \mu_n(G) \) exists.

Remark. Theorem 2 was first proved by Dieudonné [2] for \( X \) metric and later by Grothendieck for \( X \) an arbitrary compact Hausdorff space.

3. Main results.

Definition. Let \( \mathcal{B} \) be a family of Borel sets of \( X \).

(a) We call \( \mathcal{B} \) a weak converging class for \( M(X) \) provided every sequence \( \{\mu_n\} \) in \( M(X) \) which converges for each member \( E \) of \( \mathcal{B} \) (i.e. \( \{\mu_n(E)\} \) is a convergent sequence) converges for the weak topology.

(b) We call \( \mathcal{B} \) a bounding class for \( M(X) \) provided every sequence \( \{\mu_n\} \) in \( M(X) \) which is bounded on each member \( E \) of \( \mathcal{B} \) (i.e. \( \sup_n |\mu_n(E)| < \infty \)) is such that its sequence of norms \( \{||\mu_n||\} \) is bounded.

Theorem 2 states that the family \( \mathcal{B} \) of open sets of \( X \) is a weak converging class. Dieudonné also proved that it is a bounding class.

Our first theorem gives a set of sufficient conditions on a family of open sets that it be both a weak converging class and a bounding class. We will apply our theorem to show that the regular open sets form a weak converging and a bounding class.

Theorem 3. Let \( \mathcal{B} \) be a family of open sets of a compact Hausdorff space \( X \), and let \( \mathcal{B} \) satisfy

1. \( \mathcal{B} \) is a basis for the topology of \( X \);
2. If \( E_1 \) and \( E_2 \) are in \( \mathcal{B} \), then \( E_1 \cap E_2 \) is in \( \mathcal{B} \);
3. If \( E_1 \) and \( E_2 \) are in \( \mathcal{B} \), and \( E_1 \cap E_2 = \emptyset \), then \( E_1 \cup E_2 \) is in \( \mathcal{B} \).
4. If \( K \) is compact, and \( U \) is open, and \( K \subseteq U \), then there exists an \( E \) in \( \mathcal{B} \) such that \( K \subseteq E \subseteq \overline{E} \subseteq U \);
5. If \( \{E_n\} \) and \( \{G_n\} \) are sequences from \( \mathcal{B} \) such that \( E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \cdots \subseteq G_n \subseteq \cdots G_2 \subseteq G_1 \), then there is some \( E_0 \) in \( \mathcal{B} \) such that \( E_n \subseteq E_0 \subseteq G_n \) for every \( n \) (\( E_0 \) is said to interpolate the sequences); then \( \mathcal{B} \) is a weak converging family for \( M(X) \).

First we need the following

Lemma 1. Let \( \{E_n\} \) be a sequence from a family of open sets \( \mathcal{B} \) satisfying conditions (1)–(5), and suppose that \( \text{Cl}(U_{i \in A} E_i) \cap E_n = \emptyset \) for each \( n \). If \( \nu \) is any nonnegative regular Borel measure, then for every \( \delta > 0 \) there is an infinite set \( A \) of positive integers and an \( E_A \subseteq \mathcal{B} \) such that \( \bigcup_{i \in A} E_i \subseteq E_A \) and \( \nu(E_A) < \delta \).
PROOF. Let \( \delta > 0 \) be given. We begin by choosing for each \( n \) an open set \( U_{2n} \supseteq \overline{E}_{2n} \) such that \( U_{2n} \cap \text{Cl} (\bigcup_{i=1}^{\infty} E_{2i+1}) = \emptyset \). Property (4) allows us to pick for each \( n \) a set \( F_{2n} \subseteq \emptyset \) such that \( (\overline{E}_{2n})' \supseteq F_{2n} \supseteq U_{2n}' \).

Note that

\[
E_1 \subseteq E_1 \cup E_2 \subseteq E_1 \cup E_3 \cup E_5 \subseteq \cdots \subseteq \cdots \subseteq F_2 \cap F_4 \cap F_6 \subseteq F_2 \cap F_4 \subseteq F_2.
\]

By (5) there is some \( G_1 \subseteq \emptyset \) which interpolates the sequences; \( G_1 \) has the properties that \( G_1 \subseteq \bigcup_{i=1}^{\infty} E_{2i+1} \) and \( G_1 \cap \bigcup_{i=1}^{\infty} E_{2i} = \emptyset \).

If now \( \nu(G_1) < \delta \) we are done. If \( \nu(G_1') < \delta/2 \) then, since \( G_1' \) is compact and \( \nu \) is a regular measure, it follows from (4) that there is some \( F \subseteq \emptyset \) such that \( F \supseteq G_1' \supseteq \bigcup_{i=1}^{\infty} E_{2i} \), and \( \nu(F \cap G_1') < \delta/2 \). Hence

\[
\nu(F) = \nu(F \cap G_1') + \nu(F \cap G_1) = \nu(G_1') + \nu(F \cap G_1) < \delta/2 + \delta/2 = \delta.
\]

and we would be done.

If neither \( \nu(G_1) < \delta \) nor \( \nu(G_1') < \delta/2 \), then we may repeat the above process to find disjoint subsequences \( \{E_{2n}\} \) and \( \{E_{2m}\} \) of the sequence \( \{E_{2n+1}\} \) and a \( G_2 \) in \( \emptyset \) such that \( G_2 \supseteq \bigcup_{i=1}^{\infty} E_{2i} \) and \( G_2 \cap \bigcup_{i=1}^{\infty} E_{2i} = \emptyset \), and \( G_2 \subseteq G_1 \). If now \( \nu(G_2) < \delta \) we would be done. If \( \nu(G_1 \cap G_2') < \delta/2 \), then we may pick an \( H \subseteq \emptyset \) such that \( H \supseteq G_2' \) and \( \nu(H \cap G_2) < \delta/2 \). Note that

\[
G_1 \cap H = (G_1 \cap G_2') \cup (G_2 \cap H);
\]

hence

\[
\nu(G_1 \cap H) = \nu(G_1 \cap G_2') + \nu(G_2 \cap H) < \delta/2 + \delta/2 = \delta.
\]

We would be done since \( G_1 \cap H \subseteq \emptyset \) by (2), and \( G_1 \cap H \supseteq \bigcup_{i=1}^{\infty} E_{2i} \).

If neither \( \nu(G_2) < \delta \) nor \( \nu(G_1 \cap G_2') < \delta/2 \), then the above process may be repeated to get a \( G_3 \subseteq \emptyset \), \( G_3 \subseteq G_2 \). If this process does not terminate we would be able to find a decreasing sequence \( \{G_n\} \) in \( \emptyset \) with the property that \( \nu(G_1) \geq \delta \), \( \nu(G_1') \geq \delta/2 \), \( \nu(G_2) \geq \delta \), \( \nu(G_1 \cap G_2') \geq \delta/2 \); \( \cdots \); \( \nu(G_n) \geq \delta \), \( \nu(G_{n-1} \cap G_n') \geq \delta/2 \); \( \cdots \). However, the members of the sequence of sets \( G_1', G_1 \cap G_1', G_2 \cap G_1', \cdots \) are pairwise disjoint. This would imply that the total variation of \( \nu \) is infinite, which is a contradiction.

PROOF OF THEOREM 3. Let \( \{\mu_n\} \) be a sequence of regular Borel measures converging on each member of \( \emptyset \). To show that \( \{\mu_n\} \) is a Cauchy sequence for the weak topology, it would suffice to show that \( \{\mu_n - \sum_{n} \mu_{n+p}\} \) converges to 0 in the weak topology for each sequence
Theorem 1.1 it follows that there is a sequence \( \{ E_n \} \) from \( \mathfrak{B} \) and a positive \( \epsilon \) such that \( \text{Cl}(\bigcup_{n=1}^{\infty} E_n) \cap E_n = \emptyset \) for each \( n \), such that a subsequence of \( \{ \mu_n \} \), without loss of generality still called \( \{ \mu_n \} \), satisfies \( |\mu_n(E_n)| > \epsilon > 0 \).

We now carry out an inductive process to obtain a subsequence \( \{ E_{n_1} \} \) of \( \{ E_n \} \) such that \( \{ \mu_n \} \) does not converge to zero on some \( E_0 \in \mathfrak{B} \) such that \( E_0 \supseteq \bigcup_{n=1}^{\infty} E_n \). This will contradict the hypothesis that \( \{ \mu_n \} \) converges to zero on every member of \( \mathfrak{B} \).

We apply the lemma to the measure \( |\mu_1| \) to get an infinite set of positive integers \( A_1 \) and an \( E_{A_1} \in \mathfrak{B} \) such that \( \bar{E}_{A_1} \cap \bar{E}_1 = \emptyset \) and \( E_{A_1} \supseteq \bigcup_{n=1}^{\infty} E_n \) and \( |\mu_1(E_{A_1})| < \epsilon/3. \)

First set \( n_0 = 1 \) and pick \( n_1 \in A_1 \) so large that \( |\mu_n(E_1)| < \epsilon/3 \) for all \( n \geq n_1 \). Next apply the lemma again along with property (2) to extract an infinite set of positive integers \( A_2 \) from the set \( \{ A_1 \cap \text{all integers } \geq n_1 \} \) and obtain an \( E_{A_2} \subseteq E_{A_1} \), \( E_{A_2} \in \mathfrak{B} \), such that \( E_{A_2} \supseteq \bigcup_{n=1}^{\infty} E_n \), \( E_{A_2} \cap E_{A_1} = \emptyset \), and \( |\mu_1(E_{A_2})| < \epsilon/3. \)

Now pick \( n_2 > n_1, n_2 \in A_2 \), so large that \( |\mu_n(E_1)| + |\mu_n(E_{A_1})| < \epsilon/3 \) for all \( n \geq n_2 \). Continuing in this fashion we obtain a sequence of integers \( \{ n_0, n_1, n_2, \ldots \} \) and a decreasing sequence of sets in \( \mathfrak{B} \), \( E_{A_1} \supseteq E_{A_2} \supseteq E_{A_3} \supseteq \cdots \) such that the following hold:

(a) \( |\mu_{n_i}|(E_{A_{i+1}}) < \epsilon/3 \) for all \( i \),

(b) \( \sum_{i=0}^{j-1} |\mu_n(E_{n_i})| < \epsilon/3 \) for all \( n \geq n_j \).

Consider the sequence of sets:

\[
E_1 \subseteq E_1 \cup E_{n_1} \subseteq E_1 \cup E_{n_1} \cup E_{n_2} \subseteq \cdots \\
\cdots \subseteq E_{A_2} \cup E_{n_2} \cup E_{n_1} \cup E_1 \subseteq E_{A_3} \cup E_{n_1} \cup E_1 \subseteq E_{A_1} \cup E_1.
\]

Each member of the sequence is in \( \mathfrak{B} \), and by property (5) we may choose an \( E_0 \) from \( \mathfrak{B} \) which interpolates the sequence.

It simply remains to note that:

\[
|\mu_{n_j}(E_0)| \geq |\mu_{n_j}(E_{n_j})| - \sum_{i=1}^{j-1} |\mu_{n_j}(E_{n_i})| - |\mu_{n_j}|(E_{A_{j+1}}) \\
\geq \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3
\]
holds for every $j$. However, this contradicts our earlier assumption that the sequence $\{\mu_n(E)\}$ converges to 0 for each member $E$ of $\emptyset$. Q.E.D.

**Corollary.** If $\emptyset$ is a family of open sets of a compact Hausdorff space $X$ satisfying conditions (1)-(5) then $\emptyset$ is a bounding class for $M(X)$.

**Proof.** Suppose $\{\mu_i\}$ is a sequence in $M(X)$ satisfying $\sup_i |\mu_i(E)| < \infty$ for every $E \in \emptyset$. If $\{|\mu_i|\}$ is not bounded then without loss of generality we may drop to a subsequence and assume $\lim_i |\mu_i| = \infty$. Now we may multiply each $\mu_i$ by an appropriate scalar (e.g. $||\mu_i||^{1/2}/||\mu_i||$) to insure that $\lim_i \mu_i(E) = 0$ for every $E \in \emptyset$ while maintaining $\lim_i |\mu_i| = \infty$. The proof of the theorem shows that $\{\mu_i\}$ is $\sigma(M(X), M^*(X))$—convergent to zero. However, this is impossible in view of $\lim_i |\mu_i| = \infty$. Q.E.D.

**Definition.** An open set $U$ is called regular if $U = \text{int}(U)$.

The set of regular open sets of a topological space when ordered by set inclusion is a complete Boolean algebra. The supremum of a family $(U_\alpha)_{\alpha \in A}$ of regular open sets, denoted by $\bigvee_{\alpha \in A} U_\alpha$, is defined to be $\text{int}(\text{Cl}(U_\alpha \subseteq U))$, the infimum, denoted by $\bigwedge_{\alpha \in A} U_\alpha$, is defined to be $\text{int}(\text{Cl}(\bigcap_{\alpha \in A} U_\alpha))$. The intersection of two regular open sets is regular. However, the union of two regular open sets need not be regular, and this fact presents the essential difficulty, since a Borel measure need not be even finitely additive with respect to the Boolean operations. However, if the closures of two regular open sets are disjoint then their union is regular. A complete discussion of regular open sets may be found in Halmos [5, p. 13].

**Theorem 4.** If $X$ is a compact Hausdorff space and $\emptyset$ is the Boolean algebra of all the regular open sets of $X$, then $\emptyset$ is both a weak converging class and a bounding class.

**Proof.** $\emptyset$ obviously satisfies conditions (1)-(4) of Theorem 3. Also if $E_1 \subseteq E_2 \subseteq E_3 \cdots \subseteq G_3 \subseteq G_2 \subseteq G_1$ is such that each member of the sequence is a regular open set, then both $V_{1}^{\infty} E_i$ and $\Lambda_{1}^{\infty} G_i$ interpolate the sequence. Thus $\emptyset$ satisfies the conditions of Theorem 3, and we conclude that $\emptyset$ is both a weak converging class and a bounding class. Q.E.D.

**Remark.** W. G. Bade and P. C. Curtis had previously shown (unpublished) that the regular open sets are a bounding class.

**Definition.** A compact Hausdorff space $X$ is called Stonian if the closure of every open set is open.
Lemma 2. $X$ is Stonian iff every regular open set is open and closed.

Proof. If $X$ is Stonian and $U$ is a regular open set, i.e., $U = \text{int}(U)$, then $\text{int}(U) = U$ since $U$ is an open set. Hence $U = U$ and $U$ is open and closed. Conversely, if every regular open set is open and closed, then if $U$ is an open set, since $U \subseteq \text{int}(U)$, it follows that $U \subseteq \text{int}(U)$ and hence that $U = \text{int}(U)$. Thus $U$ is an open set. Q.E.D.

Our Theorem 4 is a generalization to arbitrary compact Hausdorff spaces of the following theorem of A. Grothendieck [4, p. 168].

Theorem 5. Let $X$ be Stonian and $\{\mu_n\}$ a sequence in $M(X)$. Then $\{\mu_n(E)\}$ is a convergent sequence for every open closed $E$ iff $\{\mu_n\}$ is convergent for the $\sigma(M(X), M(X)^*)$ topology.

Proof. By Lemma 2 the regular open sets of $X$ are precisely the open closed sets. Hence Theorem 4 gives us the result. Q.E.D.

Notation and Definitions. Let $S$ be a discrete set. Then $\beta S$ denotes the Stone-Čech compactification of $S$. It is well known that $\beta S$ is a Stonian space. The space of all bounded real or complex-valued functions on $S$ with the supremum norm will be denoted by $B(S)$. The space of finitely additive measures on the field $\Sigma$ of all subsets of $S$ will be denoted by $ba(S, \Sigma)$. If $\mu \in ba(S, \Sigma)$ then $||\mu|| = |\mu|(S)$, the total variation of $\mu$ on $S$. The atomic part of $\mu$ is defined by $\nu(E) = \sum_{s \in E} \mu(s)$ where $E \subseteq \Sigma$. We shall need the facts that $C(\beta S)$ is isometrically isomorphic to $B(S)$ and $M(\beta S)$ is isometrically isomorphic to $ba(S, \Sigma)$. For a complete discussion of these facts see Dunford and Schwartz [3, p. 311-313].

Grothendieck’s proof of Theorem 5 was based on the following result due to Phillips [7].

Theorem 6. Let $S$ be a discrete set and $\{\mu_n\}$ a sequence in $ba(S, \Sigma)$. If $\{\mu_n(E)\}$ converges to 0 for each $E \subseteq \Sigma$, then $||\nu_n||$ converges to 0, where $\nu_n$ is the atomic part of $\mu_n$.

Remark (Bade). Theorem 5 is equivalent to Theorem 6.

Proof. Assume Theorem 5 and that $\{\mu_n\}$ is a sequence in $ba(S, \Sigma)$ such that $\lim_n \mu_n(E) = 0$ for every $E \subseteq \Sigma$. Let $\bar{\mu}_n$ be the correspondent of $\mu_n$ in $M(\beta S)$, and $k_E$ be the correspondent of $k_E$ in $C(\beta S)$ ($k_E$ denotes the characteristic function of $E$). Then $\bar{\mu}_n(k_E) = \mu_n(k_E)$, and it follows that $\bar{\mu}_n$ converges to 0 for each open closed set in $\beta S$. By Theorem 5 $\{\bar{\mu}_n\}$ converges to 0 in the $\sigma(M(\beta S), M(\beta S)^*)$ topology. Thus $\{\mu_n\}$ converges to 0 for the $\sigma(ba(S, \Sigma), ba(S, \Sigma)^*)$ topology.

Let $P$ denote the projection of norm 1 of $ba(S, \Sigma)$ onto $l_1(S)$ defined
by $P: \mu \rightarrow \nu$ where $\nu$ is the atomic part of $\mu$. $P$ is norm continuous and hence is continuous for the weak topologies. Thus $\{P\mu_n\} = \{\nu_n\}$ converges to 0 for the $\sigma(l_1(S), l_\infty(S))$ topology. By a theorem of Banach [1, p. 137] $\{\|\nu_n\|\}$ converges to zero.

Assume Theorem 6 and that $\{\mu_n\}$ is a sequence of regular Borel measures on a Stonian space $S$ which converges to 0 on each open closed subset of $S$. To show that $\{\mu_n\}$ is weakly convergent to 0, it suffices to show (by Theorem 1) that $\{\mu_n(E_n)\}$ converges to 0, where $\{E_n\}$ is an arbitrary sequence of pairwise disjoint open closed subsets of $S$.

Define for each $n$ a set function $\nu_n$ on $N$, the set of positive integers:

$$\nu_n(A) = \mu_n\left( \bigvee_{i \in A} E_i \right) \text{ where } A \subseteq N.$$  

Note that $\nu_n$ is bounded and finitely additive, and hence an element of $ba(N, \Sigma)$. Since $\{\mu_n\}$ converges to 0 on each open closed subset of $S$, $\{\nu_n(A)\}$ converges to 0 for each $A \in \Sigma$. Theorem 6 allows the conclusion that $\lim_n \sum_{i=1}^{\infty} |\nu_n(i)| = 0$. In particular

$$\lim_n |\nu_n(n)| = \lim_n |\mu_n(E_n)| = 0. \quad \text{Q.E.D.}$$

The proof of Theorem 4 thus provides a new proof of Theorem 6.

**References**


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