A LOWER BOUND FOR THE DIMENSION OF CERTAIN $G_δ$ SETS IN COMPLETELY NORMAL SPACES

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Nagami and Roberts have proved [3, Theorem 1] that if $X$ is a normal space of dimension at least $n$ satisfying certain conditions, then $\dim (X - \bigcup_{i=1}^{n} A_i) \geq n - 1$ if the $A_i$ are disjoint closed subsets of $X$. In this paper we allow the $A_i$ to intersect provided that the dimension of the pairwise intersections is known. (The dimension of the remainder is reduced accordingly.)

By dimension (denoted $\dim$) we mean the covering dimension. The symbols $\text{bdy}$ and $\text{int}$ are used to denote the topological boundary and interior. We will say that a $T_1$ space $X$ is $n$-solid ($n$ a positive integer) if for every point $x$ of $X$ and every open neighborhood $U$ of $x$ there is a connected open neighborhood $V$ of $x$ such that $V \subseteq U$, $V$ is compact, $\dim V \geq n$, and no $(n-2)$-dimensional subset of $V$ separates it. Such a space is locally compact and locally connected. This property is hereditary on open subsets.

**Theorem 1.** Let $m$ and $n$ be integers such that $m \geq -1$ and $n \geq 1$. Let $X$ be a compact completely normal space with $\dim X \geq n$. Let $A_1, A_2, \cdots$ be a countable sequence of closed subsets of $X$ such that

1. $\dim A_i \leq n - 1$ for every integer $i$,
2. $\dim (A_i \cap A_j) \leq m$ if $i \neq j$.

Then $\dim (X - \bigcup_{i=1}^{n} A_i) \geq n - m - 2$.

**Proof.** The proof of the present theorem is similar to that of Theorem 1 in [3]. That proof has five steps:

1. $X$ is assumed to be a closed set with a certain minimal property.
2. A function $h$ from $X$ into $I^n$ is obtained.
3. An $n$-dimensional pyramid is constructed in $I^n$.
4. Assuming the theorem is false, the inverse images in $X$ of the $(n-1)$ pairs of opposite faces of the pyramid are separated by $n - 1$ closed sets whose intersection contains no continuum hitting the inverse images of both the base and the apex of the pyramid.
5. The pyramid is truncated and the inverse images of the $n$ pairs of opposite faces of the lower portion are shown not to be a defining

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system [3, Definition 5], in $X$. This is shown to be false, and a contradiction results.

The only change required in order to prove our present theorem is the substitution of the following modified step four:

Fourth step. Suppose that $\dim (X - U_{i=1}^n A_i) < n - m - 2$. Then $\dim (X - (U_{i=1}^n A_i \cup h^{-1}(p)))$ is less than $n - m - 2$ since $h^{-1}(p)$ is a $G_4$. By Lemma 1 of [3] there exist sets $B_1, \ldots, B_{n-m-2}$ which are closed in $X - h^{-1}(p)$ and such that

1. $B_i$ separates $C_i$ from $C_i'$, $1 \leq i \leq n - m - 2$,
2. $\bigcap_{i=1}^{n-m-2} B_i \subseteq U_{i=1}^n A_i$.

Let $A = \bigcup_{i \neq j} (A_i \cap A_j)$; then $A$ is an $F_\sigma$ and $\dim A \leq m$. By [2, Theorem 3.4], there exist $m+1$ sets $B_{n-m-1}, \ldots, B_{n-1}$ which are closed in $X - h^{-1}(p)$ and such that

1. $B_i$ separates $C_i'$ from $C_i$, $n - m - 1 \leq i \leq n - 1$,
2. $\bigcap_{i=n-m-1}^{n-1} B_i \subseteq X - A$.

Let $H = \bigcap_{i=1}^{n-m-1} B_i \subseteq U_{i=1}^n A_i - A$. If $H \neq \emptyset$ and $H \cap h^{-1}(B)$ and $h^{-1}(p)$ are disjoint closed subsets of $H$. Recall from step 2 that $h^{-1}(p) \subseteq X - U_{i=1}^n A_i$. Then by the above $H = h^{-1}(p) \cup (H \cap A_1) \cup (H \cap A_2) \cup \cdots$, and these pieces are disjoint and closed $(H \cap A_1 = H \cap A_i)$. As in step 4 of [3, Theorem 1], there is no continuum $K \subseteq H$ which intersects both $h^{-1}(p)$ and $H \cap h^{-1}(B)$.

Step 5 now proceeds unchanged, and the proof is complete.

Theorem 1 of [3] is obtained as a special case of this theorem when $m = -1$.

**Theorem 2.** If $X$ is a connected $n$-solid space, then $X$ is not the countable union of closed proper subsets $A_1, A_2, \ldots$ such that $\dim (A_i \cap A_j) \leq n - 2$ when $i \neq j$.

**Proof.** Suppose the theorem is false, and $X = \bigcup_{i=1}^\infty A_i$, where the $A_i$ satisfy the conditions of the theorem. We will construct a sequence $\overline{V}_0, \overline{V}_1, \cdots$ of compact nonempty subsets of $X$ such that for every integer $i$, $\overline{V}_{i+1} \subseteq \overline{V}_i$ and $\overline{V}_i \cap A_i = \emptyset$. This is a contradiction since the $\overline{V}_i$ have the finite intersection property and an empty total intersection.

By Baire's theorem there is an $i$ such that $\text{int} (A_i)$ is not empty. Fix $i$ at this value. Then $\text{bdy} (\text{int}(A_i)) \neq \emptyset$ since $X$ is connected. Let $x_0 \in \text{bdy}(\text{int}(A_i))$. Let $x_0 \in \overline{V}_0 \subseteq \overline{V}_0 \subseteq X$ be as in the definition of $n$-solid. Then $\overline{V}_0$ is not a subset of $A_i$ since $x_0$ is not in $\text{int}(A_i)$. Suppose there is an integer $j \neq i$ such that $\overline{V}_0 \subseteq A_j$. Then $A_j \cap A_i \subseteq \overline{V}_0 \cap \text{int}(A_i)$, a nonempty open set. Hence $A_j \cap A_i$ contains a closed set of dimension at least $n$ by the definition of $n$-solid, so $\dim (A_i \cap A_j)$

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$\geq n$, a contradiction. Then $x_0$ and $V_0$ satisfy (1)–(4) below if we let $V_\infty = X$ and $A_0 = \emptyset$.

**Induction step.** Suppose we are given $x_p$ and $V_p$ such that

1. $x_p \in V_p$, and $\overline{V}_p$ is as in the definition of $n$-solid;
2. for every integer $j$, $\overline{V}_p \subseteq A_j$;
3. $\overline{V}_p \subseteq \overline{V}_{p-1}$;
4. $\overline{V}_p \cap A_p = \emptyset$.

We will construct $x_{p+1}$ and $V_{p+1}$ satisfying (1)–(4).

Let $W$ be the open set $V_p - A_{p+1}$. If $W$ is connected, then it is not a subset of any $A_j$, since otherwise $\overline{V}_p \subseteq A_j \cup A_{p+1}$, so $A_j \cap A_{p+1}$ separates $\overline{V}_p$, a contradiction.

If $W$ is not connected, let $K$ be a component; $K$ is open. Let $L = \text{int}(\overline{K})$, so $K \subseteq L \subseteq \overline{L} = K$. But some other component of $W$ is not empty and $V_p$ is connected. Therefore $\text{bdy}(L) \cap V_p$ is nonempty, separates $V_p$, and is a subset of $A_{p+1}$.

There is no integer $j$ such that $K \subseteq A_j$, for suppose there is such a $j$. Then $\overline{K} \subseteq A_j$, so $\text{bdy}(L) \cap V_p \subseteq A_j \cap A_{p+1}$. Let $x \in \text{bdy}(L) \cap V_p$ and let $x \in V \subseteq \overline{V} \subseteq V_p$ be as in the definition of $n$-solid. Then $V$ is not a subset of $\overline{L}$ since $x$ is not in $L = \text{int}(\overline{L})$. Also $V \cap L \neq \emptyset$ since $x$ is a limit point of $L$. Therefore $\text{bdy}(L) \cap V$ separates $V$, a contradiction.

In either case we get a connected open subset ($K$ or $W$) of $X$ which is not a subset of any $A_j$. Since this set ($K$ or $W$) is $n$-solid, we can repeat the process used to get $x_0$ and $V_0$. This gives us the $x_{p+1}$ and $V_{p+1}$ we want, completing the induction and the proof.

It is well known that no continuum is a countable union of disjoint closed subsets. In particular, no locally connected continuum is such a union. Since a locally connected continuum is 1-solid, Theorem 2 is a direct generalization of this special case.

Nagami and Roberts have given an example [3, Figure 2] which shows that the conclusion of Theorem 2 does not hold if we require only that $X$ be a Cantor $n$-manifold. This example is a Cantor 2-manifold which is the countable union of disjoint closed sets $A_{i,j}$ and a Cantor discontinuum.

**Corollary.** If $X$ is Hausdorff, connected, locally connected, and locally compact, then $X$ is not a countable union of disjoint closed subsets.

**Proof.** Such a space is 1-solid.

Sierpinsky has given an example (see [1, p. 188]) which shows that local connectedness is not superfluous in the Corollary.

**Theorem 3.** Let $m$ and $n$ be integers such that $m \geq -1$ and $n \geq 1$. If $X$ is connected, and every point in $X$ has a neighborhood homeomorphic
to the n-cube $I^n$, and $A_1$, $A_2$, \ldots are closed proper subsets of $X$ such that $\dim (A_i \cap A_j) \leq m$ whenever $i \neq j$, then $\dim (X - \bigcup_{i=1}^m A_i) \geq n - m - 2$.

**Proof.** $X$ is connected and $n$-solid. By Theorem 2 there is a point $x \in X - \bigcup_{i=1}^m A_i$. There is a neighborhood $F$ of $x$ which is homeomorphic to $I^n$. The dimension of $X - \bigcup_{i=1}^m A_i$ is at least as great as the dimension of $F - \bigcup_{i=1}^m A_i$, so it is sufficient to prove the theorem for the case where $X = I^n$.

Let $S^{n-1}$ denote the surface of $I^n$. Then $(I^n - S^{n-1})$ is connected and $n$-solid, and no $A_i$ contains all of it. By Theorem 2 there is a point $q \in (I^n - S^{n-1}) - \bigcup_{i=1}^m A_i$.

Recall step 3 of the proof of Theorem 1 in [3]. A pyramid with apex $p$ and base $B$ has been inscribed in $I^n$. Let $h$ be a homeomorphism from $I^n$ onto $I^n$ which leaves $S^{n-1}$ fixed and such that $h(q) = p$. Then $h^{-1}(p) \cap (\bigcup_{i=1}^m A_i) = \emptyset$ as before. Let $f = h|S^{n-1}$, the identity map. Now, using these mappings $h$ and $f$, repeat step 3, the modified step 4, and step 5 to complete the proof.

Corollary 2 of [3] is the special case of this theorem obtained when $m = -1$.

**References**


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