AN EQUIVALENCE THEOREM FOR EMBEDDINGS OF
COMPACT ABSOLUTE NEIGHBORHOOD RETRACTS

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In this paper we wish to prove the following theorem.

**Theorem 1.** Suppose that each of $A_0$ and $A_1$ is a compact absolute neighborhood retract (ANR) of dimension $k$ in Euclidean $n$-space $E^n$ $(2k+2 \leq n, n \geq 5)$ such that $E^n - A_i$ is 1-ULC (uniformly locally simply connected) for $i = 0, 1$, and $f: A_0 \to A_1$ is a homeomorphism such that $d(a, f(a)) < \varepsilon$ for each $a \in A_0$. Then there exists an $\varepsilon$-push $h$ of $(E^n, A^0)$ such that $h|A_0 = f$.

In [2] the authors showed that if $A$ is a $k$-dimensional polyhedron topologically embedded in $E^n$ $(2k+2 \leq n, n \geq 5)$ such that $E^n - A$ is 1-ULC, then for each $\varepsilon > 0$, there is an $\varepsilon$-push $h$ of $(E^n, A)$ such that $h|A: A \to E^n$ is piecewise linear. Hence, a well-known theorem of Bing and Kister [1, Theorem 5.5] applies to prove Theorem 1 when $A_0$ is a polyhedron. In fact, Theorem 5.5 of [1], together with the techniques of Homma [4] and Gluck [3] and the following engulfing theorem proved in [2], make our result possible.

**Theorem 2.** Suppose that $A$ is a $k$-dimensional compact ANR in $E^n$ $(n-k \geq 3, n \geq 5)$ such that $E^n - A$ is 1-ULC and $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $f: A \to E^n$ is a $\delta$-homeomorphism and $U$ is an open subset of $E^n$ containing $f(A)$, then there exists an $\varepsilon$-push $h$ of $(E^n, A)$ such that $h(U) \supset A$.

Following Gluck [3], we define an $\varepsilon$-push $h$ of the pair $(X, A)$, where $X$ is a metric space and $A$ is a subset of $X$ such that $A$ is compact, to be a homeomorphism of $X$ onto itself that is $\varepsilon$-isotopic to the identity by an isotopy $h_t$ $(t \in [0, 1])$ of $X$ such that for each $t \in [0, 1]$, $h_t|X - N_t(A) = \text{identity}$. Other terminology used here is standard, and we shall assume that it is familiar to the reader.

Actually, the proof of Theorem 1 follows from known results, once we prove a

**Lemma.** Suppose that $A$ is a compact ANR of dimension $k$ in $E^n$ $(2k+2 \leq n, n \geq 5)$ such that $E^n - A$ is 1-ULC and $f: A \to E^n$ is an embedding such that $d(a, f(a)) < \varepsilon$ for each $a \in A$. Then for each $\delta > 0$ there exists an $\varepsilon$-push $h$ of $(E^n, A)$ such that $d(h(a), f(a)) < \delta$ for each $a \in A$. 

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Theorem 1 is then proved exactly as is Theorem 4.4 of [3], and we shall not repeat the details of the constructions involved.

Proof of the Lemma. Given \( \delta > 0 \), there exists \( \eta > 0 \) such that if \( a, b \in A \) with \( d(a, b) < \eta \), then \( d(f(a), f(b)) < \frac{1}{\delta} \delta \). Let \( N \) be a polyhedral neighborhood of \( A \) in \( E^n \) that retracts onto \( A \) by a retraction \( r: N \rightarrow A \) such that \( d(x, r(x)) < \frac{1}{\eta} \eta \) and \( d(x, fr(x)) < \varepsilon \) for each \( x \in N \). Let \( T \) be a triangulation of \( N \) with mesh less than \( \frac{1}{\eta} \eta \) and let \( T^k \) denote the \( k \)-skeleton of \( T \), with \( N^k = |T^k| \), the polyhedron of \( T^k \).

The mapping \( fr: N^k \rightarrow E^n \) has the property that if \( x \in N^k, a \in A, \) and \( d(x, a) < \frac{1}{\eta} \eta \), then \( d(fr(x), f(a)) < \frac{1}{\eta} \delta \), since \( d(r(x), a) < \frac{1}{\delta} \delta \).

Let \( g': N^k \rightarrow E^n \) be a piecewise linear embedding such that \( d(g'(x), fr(x)) < \frac{1}{\eta} \delta \) and \( d(x, g'(x)) < \varepsilon \) for each \( x \in N^k \). Since \( 2k + 2 \leq n \), we may apply Theorem 5.5 of [1] to obtain an \( \varepsilon \)-push \( g \) of \( (E^n, N^k) \) such that \( g|N^k = g' \). Notice that if \( x \in N^k, a \in A, \) and \( d(x, a) < \frac{1}{\eta} \eta \), then \( d(g(x), f(a)) \leq d(g(x), fr(x)) + d(fr(x), f(a)) < \delta \). Thus there is an open set \( U \) in \( E^n \) containing \( N^k \) such that the above implication is true for \( x \in U \); that is, if \( x \in U, a \in A, \) and \( d(x, a) < \frac{1}{\eta} \eta \), then \( d(g(x), f(a)) < \delta \).

We need one additional fact concerning the open set \( U \).

Sublemma. There exists a \( \frac{1}{2} \eta \)-push \( \phi \) of \( (E^n, A) \) such that \( \phi(A) \subseteq U \).

Proof. Let \( T^{n-k-1} \) be the dual \((n-k-1)\)-skeleton of \( T \) with \( \tilde{N}^{n-k-1} = |T^{n-k-1}| \). Choose \( \eta' > 0 \) corresponding to \( \frac{1}{2} \eta \) as in Theorem 2. From the construction in the proof of Theorem 5.5 of [5], we can obtain an embedding \( \psi \) of \( A \) into \( E^n \) such that \( d(a, \psi(a)) < \eta' \) for each \( a \in A \) and \( \psi(A) \cap \tilde{N}^{n-k-1} = \emptyset \). Let \( V \) be an open subset of \( E^n \) containing \( \psi(A) \) such that \( V \cap \tilde{N}^{n-k-1} = \emptyset \).

By Theorem 2, there exists a \( \frac{1}{2} \eta \)-push \( \phi_1 \) of \( (E^n, A) \) such that \( \phi_1(V) \supseteq A \). Then \( \phi_1^{-1} \) is a \( \frac{1}{2} \eta \)-push of \( (E^n, A) \) and \( \phi_1^{-1}(A) \cap \tilde{N}^{n-k-1} = \emptyset \). Since the mesh of \( T \) is less than \( \frac{1}{2} \eta \), the technique of Stallings [7] may be used to obtain a \( \frac{1}{2} \eta \)-push \( \phi \) of \( (E^n, \phi_1^{-1}(A)) \) such that \( \phi \circ \phi_1^{-1}(A) \subseteq U \).

We complete the proof of the Lemma by setting \( h = g \phi \). We may assume that \( \eta \) is chosen sufficiently small so that the composition \( g \phi \) is an \( \varepsilon \)-push of \( (E^n, A) \). Given \( a \in A \), we have \( d(\phi(a), a) < \frac{1}{\eta} \eta \) and \( \phi(a) \in U \), so that \( d(g(\phi(A), f(a)) = d(h(a), f(a)) < \delta \).

The question as to whether Theorem 1 is true when \( k = 1 \) and \( n = 4 \) seems very hard to answer. The method used to prove Theorem 1 involves engulfing techniques that are valid only for \( n \geq 5 \). It might be possible, however, to improve the codimension restriction by one if certain other conditions are satisfied. For example, Price has shown [6] that any two piecewise linear embeddings of a \( k \)-complex \( K \) into \( E^n (n = 2k+1) \) are equivalent by an isotopy of \( E^n \) that is the identity outside a compact set if \( H^k(K, Z) = 0 \). A natural question then is
Question 1. Is Theorem 1 true with \( n = 2k + 1 \) if \( H^k(A_0, Z) = 0 \)? In particular, one might consider a special case.

Question 2. Is Theorem 1 true with \( n = 2k + 1 \) if \( A_0 \) is an absolute retract?

References


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