ON A THEOREM OF FEJÉR AND RIESZ

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1. In what follows we suppose that \( r > 1 \) and that \( A_r \) is a constant, depending only on \( r \), the value of which is not usually the same at each occurrence. Let \( U(\theta) \) denote a real function measurable over \((-\pi, \pi)\), let
\[
P(p, \theta) = \frac{(1 - p^2)[(1 - \rho)^2 + 4\rho \sin^2 \frac{\theta}{2}]}{[1 - \rho^2]^2 + 4\rho^2 \sin^2 \frac{\theta}{2}} - 1, \quad 0 \leq \rho < 1,
\]
and let
\[
u(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \theta) U(\theta) d\theta.
\]
\( P(\rho, \theta) \) is the Poisson kernel and \( u(\rho) \equiv u(\rho, 0) \) is the value at the point \((\rho, 0)\) of a function \( u(\rho, \theta) \) in polar coordinates which is harmonic inside the unit disc and has boundary value \( U(\theta) \) on the unit circle.

We begin by giving a new ‘real variable’ proof of the following theorem of Fejér-Riesz type. This is similar to a proof given by du Plessis [3] but it differs in a way which leads to a new analogue in three dimensions. Before du Plessis' paper appeared, the only proof available was of a ‘complex variable’ nature and based on the Fejér-Riesz inequality \( \int_0^\infty |f(r)| dr \leq \frac{1}{2} \int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta \) [1].

**Theorem 1.**

\[
\int_0^1 |u(\rho)| r d\rho \leq A_r \int_{-\pi}^{\pi} |U(\theta)| r d\theta.
\]

The proof is based on the use of an inequality theorem (see, for example, [2, p. 229, Theorem 319]) which we state as a lemma.

**Lemma.** If \( f(x) \) is nonnegative, \( K(x, y) \) nonnegative and homogeneous of degree \(-1\) and \( \int_0^\infty K(x, 1)x^{-1/r} dx = k \), then
\[
\int_0^\infty \left( \int_0^\infty K(x, y)f(x) dx \right) dy \leq k \int_0^\infty (f(x))^{1/r} dx.
\]

**Proof of Theorem 1.** For \( 0 \leq \rho < 1 \), since \( P(\rho, \theta) > 0 \), we have
\[
|u(\rho)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \theta)|U(\theta)| d\theta,
\]

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so it is enough to prove the theorem under the assumption that 
$U(\theta) \geq 0$. We may further assume that $U(\theta)$ is an even function of $\theta$
and prove (what is then equivalent to (1)) that

$$
\int_0^1 \left( \frac{1}{\pi} \int_0^\pi P(\rho, \theta)U(\theta)d\theta \right)^r d\rho \leq A_r \int_0^\pi (U(\theta))^r d\theta,
$$

for on replacing $U(\theta)$ in (3) by the even function $U(\theta) + U(-\theta)$, and
using the Hölder inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for $a \geq 0$, $b \geq 0$, we obtain

$$
\left[ \frac{1}{2\pi} \int_0^\pi \left( \int_0^\pi P(\rho, \theta)(U(\theta) + U(-\theta))d\theta \right)^r d\rho \right] \leq A_r \int_0^\pi (U(\theta) + U(-\theta))^r d\theta \leq A_r \int_0^\pi 2^{r-1}[(U(\theta))^r + (U(-\theta))^r]d\theta = A_r \int_0^\pi (U(\theta))^r d\theta.
$$

Dividing the range of integration with respect to $\rho$ in (3) into the
two intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, we first consider integration over $(0, \frac{1}{2})$. Since $P(\rho, \theta) \leq (1+\rho)/(1-\rho)$ we have, using Hölder's inequality,

$$
\int_0^{1/2} \left( \frac{1}{\pi} \int_0^\pi P(\rho, \theta)U(\theta)d\theta \right)^r d\rho \leq \int_0^{1/2} \pi^{-r} \left( \frac{1 + \rho}{1 - \rho} \right)^r \left( \int_0^\pi U(\theta)d\theta \right)^r d\rho = A_r \left( \int_0^\pi U(\theta)d\theta \right)^r \leq A_r \int_0^\pi (U(\theta))^r d\theta.
$$

Next, defining

$$
P_1(\rho, \theta) = \frac{2(1 - \rho)}{(1 - \rho)^2 + \theta^2},
$$

for the interval $(\frac{1}{2}, 1)$, since $\frac{1}{2} \theta \geq \theta/\pi$ over $(0, \pi)$, we have

$$
P(\rho, \theta) \leq \frac{2(1 - \rho)}{(1 - \rho)^2 + 2\theta^2/\pi^2} = \frac{1}{2} \frac{2(1 - \rho)}{\frac{1}{2} \pi^2 (1 - \rho)^2 + \theta^2} < \frac{1}{2} \pi^2 P_1(\rho, \theta),
$$

and so
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\[ \int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P(\rho, \theta) U(\theta) d\theta \right)^r d\rho \]

(6)

\[ \leq A_r \int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P_1(\rho, \theta) U(\theta) d\theta \right)^r d\rho. \]

With \( f(x) = U(x) \) for \( 0 \leq x \leq \pi, f(x) = 0 \) for \( x > \pi \), and

\[ K(x, y) = (2/\pi) y/(x^2 + y^2), \]

the functions \( f(x), K(x, y) \) satisfy the conditions of the lemma (with \( k = \csc \pi/(1 - 1/r) \)). On replacing \( x \) by \( \theta \), \( y \) by \( 1 - \rho \), (2) and (5) give

(7)

\[ \int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P_1(\rho, \theta) U(\theta) d\theta \right)^r d\rho \leq \int_{-\pi}^{\pi} \leq A_r \int_{0}^{\pi} (U(\theta))^r d\theta, \]

and finally (4) and combination of (6) and (7) give the inequality (3). This completes the proof of the theorem.

2. When analogues of Theorem 1 for functions harmonic in the unit sphere are considered there are two possibilities. Let \( (\rho, \theta, \phi) \) denote spherical polar coordinates, \( U(\theta, \phi) \) a real function measurable for \( 0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi \), and let

\[ Q(\rho, \theta) = (1 - \rho^2) [(1 - \rho^2) + 4\rho \sin^2 \frac{\theta}{2}]^{-3/2} \sin \theta, \]

\[ u(\rho) = \int_{0}^{\pi} \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) Q(\rho, \theta) d\theta. \]

Here \( Q(\rho, \theta) \) is the three-dimensional Poisson kernel and \( u(\rho) \equiv u(\rho, 0, 0) \) is the value at the point \( (\rho, 0, 0) \) of a function \( u(\rho, \theta, \phi) \) harmonic inside the unit sphere and with boundary values \( U(\theta, \phi) \) on the surface. The possibilities are

(8)

\[ \int_{0}^{1} \left( \int_{0}^{r} u(r) \right)^r d\rho \leq A_r \int_{0}^{\pi} \int_{-\pi}^{\pi} |U(\theta, \phi)|^r \sin \theta d\theta d\phi. \]

and

\[ \int_{0}^{1} |u(\rho, \theta, \phi)|^r \rho d\phi d\theta \leq A_r \int_{0}^{\pi} \int_{-\pi}^{\pi} |U(\theta, \phi)|^r \sin \theta d\phi d\theta. \]

In both inequalities the right-hand side is the integral of \( |U(\theta, \phi)|^r \) over the surface of the unit sphere. In the first inequality, the left-hand side consists of two integrations of \( u(\rho, \theta, \phi) \) along a radius, in the second inequality the left-hand side is the integral of \( u(\rho, \theta, \phi) \) over a diametral plane. Both of these analogues are, in fact, valid and
they are particular cases of a general theorem of du Plessis [3] concern- 
ing functions in \( n \) dimensions. du Plessis' proof of this general 
theorem is indirect and depends on half-space analogues of Theorem 
1 [4]. In this note we give a direct proof of a stronger version of (8) 
which does not seem to be deducible using du Plessis' argument.

**Theorem 2.**

(9) \[ \int_0^1 (1 - \rho) | u(\rho) | r d\rho \leq A_r \int_0^\pi \left( \int_{-\pi}^\pi | U(\theta, \phi) | d\phi \right)^r \sin \theta d\theta. \]

The left-hand side here is identical to the left-hand side of (8) and, 
by Hölder's inequality,

\[ \int_0^\pi \left( \int_{-\pi}^\pi | U(\theta, \phi) | d\phi \right)^r \sin \theta d\theta \leq A_r \int_0^\pi \int_{-\pi}^\pi | U(\theta, \phi) |^r \sin \theta d\phi d\theta, \]

so that (9) is a stronger inequality than (8).

**Proof.** Arguing as before, it is enough to prove the theorem under 
the assumption that \( U(\theta, \phi) \equiv 0 \).

We divide the range of integration with respect to \( \rho \) as before, and 
first consider integration over \((0, \frac{1}{2})\). Since 

\[ Q(\rho, \theta) \leq (1 + \rho)(1 - \rho)^{-2} \sin \theta \]

we have, using Hölder's inequality,

\[ \int_0^{1/2} (1 - \rho) (u(\rho))^r d\rho \]

\[ = \int_0^{1/2} (1 - \rho) \left[ \frac{1}{4\pi} \int_0^{\pi} \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) Q(\rho, \theta) d\theta \right]^r d\rho \]

\[ \leq \int_0^{1/2} (1 - \rho) \left[ \frac{1 + \rho}{(1 - \rho)^{2r}} \right]^r d\rho \]

\[ = A_r \left[ \int_0^{\pi} \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) \sin \theta d\theta \right]^r \]

\[ \leq A_r \int_0^{\pi} \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right)^r \sin \theta d\theta. \]
Next, defining

\[ R(\rho, \theta) = 2(1 - \rho)^{1+1/r} - 1/r [1 - (1 - \rho)^2 + \theta^2]^{-3/2} \]  

for the interval \((\frac{1}{2}, 1)\), since \(\sin \theta \leq \theta\) and \(\frac{1}{2}\theta \geq \theta / \pi\) over \((0, \pi)\), we have

\[
Q(\rho, \theta) \leq 2(1 - \rho) [1 - (1 - \rho)^2 + 2\theta^2/\pi^2]^{-3/2} \sin \theta \\
= 2^{-1/2}\pi^3(1 - \rho) [\frac{1}{2}\pi^2(1 - \rho)^2 + \theta^2]^{-3/2} \sin \theta \\
\leq 2^{-1/2}\pi^3(1 - \rho) [1 - (1 - \rho)^2 + \theta^2]^{-3/2} \sin \theta \\
\leq 2^{-3/2}\pi^3(1 - \rho)^{-1/r} \sin^{1/r} \theta \ R(\rho, \theta),
\]

and so

\[
\int_{1/2}^1 (1 - \rho)(u(\rho))^r d\rho \\
= \int_{1/2}^1 (1 - \rho) \left[ \frac{1}{4\pi} \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) Q(\rho, \theta) d\theta \right] d\rho \\
\leq A_r \int_{1/2}^1 \left[ \frac{1}{4\pi} \int_0^\pi R(\rho, \theta) \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) \sin^{1/r} \theta \right] d\phi.
\]

Defining \(f(x) = \sin \frac{1}{r} x \int_{-\pi}^x U(x, \phi) d\phi\) for \(0 \leq x \leq \pi\), \(f(x) = 0\) for \(x > \pi\), and

\[ K(x, y) = (2\pi)^{-1} x^{1-1/r} y^{1+1/r} (x^2 + y^2)^{-3/2}, \]

the functions \(f(x), K(x, y)\) satisfy the conditions of the lemma (with \(k = \frac{1}{2}\pi^{-3/2} \Gamma(1-1/r) \Gamma(1/2+1/r)\)). On replacing \(x\) by \(\theta\), \(y\) by \(1 - \rho\), (2) and (11) give

\[
\int_{1/2}^1 \left[ \frac{1}{4\pi} \int_0^\pi R(\rho, \theta) \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) \sin^{1/r} \theta \right] d\rho \\
\leq A_r \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) \sin \theta d\theta.
\]

Combination of (12) and (13) now gives

\[
\int_{1/2}^1 (1 - \rho)(u(\rho))^r d\rho \leq A_r \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) d\phi \right) \sin \theta d\theta,
\]

and addition of (10) and (14) yields the desired inequality (9).
References


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