TWO THEOREMS OF EULER AND A GENERAL PARTITION THEOREM

GEORGE E. ANDREWS

1. Introduction. Euler proved that the partitions of a natural number \( n \) into distinct parts are equinumerous with the partitions of \( n \) into odd parts [2, p. 277]. A second theorem due to Euler states that every natural number is uniquely representable as a sum of distinct powers of 2 [2, p. 277].

If \( S_1 \) and \( S_2 \) are subsets of the natural numbers \( \mathbb{N} \), let us call \((S_1, S_2)\) an Euler-pair if for all natural numbers, \( n \), the number of partitions of \( n \) into distinct parts taken from \( S_1 \) equals the number of partitions of \( n \) into parts taken from \( S_2 \). Euler’s first theorem may then be stated by saying \( \mathbb{N} \) and \( \{ n \in \mathbb{N} | 2 \mid n \} \) are an Euler-pair, and his second theorem may be stated by saying that \( \{ 2^n | n \in \mathbb{N} \text{ or } n = 0 \} \) and \( \{ 1 \} \) are an Euler-pair. Other examples of Euler-pairs are

\[
(\{ n \in \mathbb{N} | 3 \mid n \}, \{ n \in \mathbb{N} | n = 1, 5 \text{ (mod 6)} \})
\]
due to I. J. Schur [3, p. 495], and

\[
(\{ n \in \mathbb{N} | n = 2, 4, 5 \text{ (mod 6)} \}, \{ n \in \mathbb{N} | n = 2, 5, 11 \text{ (mod 12)} \})
\]
due to H. Göllnitz [1, p. 175]. The object of this paper is to give a simple characterization of Euler-pairs.

Throughout this paper all sets \( S_i \) which we consider will be subsets of the natural numbers \( \mathbb{N} \). The notation \( mS_i \) denotes the set \( \{ mn | n \in S_i \} \). The notation \( S_i - S_j \) denotes the set \( \{ n \in S_i | n \notin S_j \} \).

By \( p(S_i; n) \) we denote the number of partitions of \( n \) into parts taken from \( S_i \). By \( q(S_i; n) \) we denote the number of partitions of \( n \) into distinct parts taken from \( S_i \). We shall write \( S_i = \{ s_1(i), s_2(i), s_3(i), \ldots \} \) where the elements are arranged in ascending order of magnitude.

**Theorem 1.** \((S_1, S_2)\) is an Euler pair if and only if \( 2S_1 \subseteq S_1 \) and \( S_2 = S_1 - 2S_1 \).

In §2 we shall prove Theorem 1. In §3 we examine some of the corollaries of Theorem 1.

2. Proof of Theorem 1. First we require the following result.

**Lemma.** If for every natural number \( n \), \( p(S_1; n) = p(S_2; n) \), then
\[ S_1 = S_2. \text{ Indeed if } m \text{ is the least integer in } (S_1 \cup S_2) - (S_1 \cap S_2), \text{ then } p(S_1; m) \neq p(S_2; m). \]

**Proof.** We need only show that the second statement is valid. Without loss of generality we assume \( m = s_r(1) \). Now \( p(S_1; m) \) is just the number of partitions of \( m \) into parts taken from the set \( \{ s_1(1), s_2(1), \ldots, s_r(1) \} \). On the other hand, \( p(S_2; m) \) is just the number of partitions of \( m \) into parts taken from \( S_2 \) which do not exceed \( m \) in magnitude; consequently by the definition of \( m \), \( p(S_2; m) \) is just the number of partitions of \( m \) into parts taken from the set \( \{ s_1(1), s_2(1), \ldots, s_{r-1}(1) \} \). Therefore \( p(S_1; m) = p(S_2; m) + 1 \). This completes the proof of the lemma.

**Proof of Theorem 1.** First we treat sufficiency. The generating function for \( q(S_1; n) \) is \( \prod_{n \in S_1} (1 + q^n) \), which is absolutely and uniformly convergent for \( \|q\| \leq 1 - \delta \).

The generating function for \( p(S_2; n) \) is \( \prod_{n \in S_2} (1 - q^n)^{-1} \), again absolutely and uniformly convergent for \( \|q\| \leq 1 - \delta \).

Now assuming \( 2S_1 \subseteq S_1 \) and \( S_2 = S_1 - 2S_1 \), we have

\[
\prod_{n \in S_1} (1 + q^n) = \prod_{n \in S_1} (1 - q^{2n})(1 - q^n)^{-1}
= \prod_{n \in S_1 - 2S_1} (1 - q^n)^{-1} = \prod_{n \in S_2} (1 - q^n)^{-1}.
\]

This establishes that \( q(S_1; n) = p(S_2; n) \) for all \( n \). Therefore \( (S_1, S_2) \) is an Euler-pair.

Next we treat necessity. We suppose that \( (S_1, S_2) \) is an Euler-pair. If \( (S_1, S_3) \) were also an Euler-pair, then by the lemma and the definition of Euler-pair we have \( S_2 = S_3 \). If we can show \( 2S_1 \subseteq S_1 \), then as above we know that \( (S_1, S_1 - 2S_1) \) is an Euler-pair and hence \( S_2 = S_1 - 2S_1 \). Thus we need only show that \( 2S_1 \subseteq S_1 \).

Suppose \( 2S_1 \not\subseteq S_1 \). Let \( s_r(1) \) be the least element of \( S_1 \) such that \( 2s_r(1) \in S_1 \). Now

\[
\prod_{n \in S_2; n < 2s_r(1)} (1 - q^n)^{-1} \prod_{n \in S_1; n \geq 2s_r(1)} (1 - q^n)^{-1}
= \prod_{n \in S_1} (1 + q^n) = \prod_{n \in S_1} (1 - q^{2n})(1 - q^n)^{-1}
= \prod_{n \in S_1; n < s_r(1)} (1 - q^{2n}) \prod_{n \in S_1; n \geq 2s_r(1)} (1 - q^n)^{-1} \prod_{n \in S_1; n \geq 2s_r(1)} (1 - q^{2n})
\cdot \prod_{n \in S_1; n > 2s_r(1)} (1 - q^n)^{-1} = \prod_{n \in S_1; n \geq 2s_r(1)} (1 - q^n)^{-1} \prod_{n \in S_1; n > 2s_r(1)} (1 - q^{2n}).
\]
Thus from the above identity we see that

\[ \prod_{n \in S_1; n < 2r, (1)} (1 - q^n)^{-1} \] and \[ \prod_{n \in S_1 - 2S_1; n < 2r, (1)} (1 - q^n)^{-1} \]

agree as power series in \( q \) for the first \( 2s_r(1) \) coefficients. However, if these two functions were unequal, then by the lemma they would have unequal coefficients among the first \( 2s_r(1) \) coefficients; hence these products are equal. Thus, cancelling them in the above equation, we obtain

\[ \prod_{n \in S_1; n < 2r, (1)} (1 - q^n)^{-1} = \prod_{n \in S_1; n < 2r, (1)} (1 - q^{2n}) \prod_{n \in S_1; n > 2r, (1)} (1 - q^n)^{-1}. \]

Consider now the coefficient of \( q^{2n(1)} \) in the power series expansion of both sides of this equation. On the left-hand side it is either 0 or 1, and on the right-hand side it is \(-1\). Thus we have a contradiction, and therefore we must have \( 2S_1 \subseteq S_1 \). This completes the proof of Theorem 1.

3. Corollaries. We start with the following special case of Theorem 1.

**Theorem 2.** Let \( S_1 = N \) be such that \( n \in S_1 \) if and only if \( 2n \in S_1 \). Let \( S_2 = \{ n \in S_1 | n \equiv 1 \pmod{2} \} \). Then \( (S_1, S_2) \) is an Euler-pair.

**Proof.** We need only show \( S_1 - 2S_1 = \{ n \in S_1 | n \equiv 1 \pmod{2} \} \).

Clearly the set on the left-hand side contains the set on the right-hand side. On the other hand, if \( n \in S_1 \) and \( n = 2m \), then by hypothesis \( m \in S_1 \), hence \( n \in 2S_1 \); consequently \( n \notin S_1 - 2S_1 \). Thus we have Theorem 2.

Both of Euler's theorems are obvious consequences of Theorem 2, as is Schur's result. Göllnitz's theorem is not a corollary of Theorem 2, but it is easily deduced from Theorem 1. We may also prove many other partition theorems which do not seem to have been noted.

**Theorem 3.** The number of partitions of a natural number \( n \) into distinct parts, each of which is representable as the sum of two squares, equals the number of partitions of \( n \) into odd parts each of which is representable as the sum of two squares.

**Proof.** By [2, p. 299], \( n \) is representable as the sum of two squares if and only if \( 2b \) is. The desired result now follows from Theorem 2.

**Theorem 4.** If \( p \) is a prime \( \equiv 1, 7 \pmod{8} \), then the number of partitions of a number \( n \) into distinct quadratic residues \( \pmod{p} \) equals the
Proof. Since \( p \equiv 1, 7 \pmod{8} \), 2 is a quadratic residue (mod \( p \)) \cite[2, p. 75]{}. Thus \( n \) is a quadratic residue (mod \( p \)) if and only if \( 2n \) is a quadratic residue (mod \( p \)). The desired result now follows from Theorem 2.

References


Pennsylvania State University