DERIVATIONS OF THE LIE ALGEBRA OF POLYNOMIALS UNDER POISSON BRACKET

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Abstract. We exhibit a class of outer derivations of the Lie algebra $P$ of complex polynomials under Poisson bracket, and prove that every derivation of $P$ is a linear combination of one of these and an inner derivation, although this decomposition may not be unique. In particular, we show that any derivation of $P$ which maps constants to zero must be inner. We use these results to characterise certain solutions of the Dirac problem.

1. Introduction. Let $E$ denote the collection of all $C^\infty$ complex-valued functions of $2n$ real variables $x=(q_1, \ldots, q_n, p_1, \ldots, p_n)$, and define the Poisson bracket of two elements of $E$ to be

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

It is well known [1] that this defines a Lie bracket on $E$, and we shall denote the corresponding Lie algebra also by $E$. The collection of all complex polynomials in the variables $x$ forms a Lie subalgebra of $E$, which we shall denote by $P$. A derivation of $P$ is a linear map $D: P \to P$ such that

$$\{D(f), g\} + \{f, D(g)\} = D(\{f, g\}) \quad \text{all } f, g \in P.$$ 

A derivation of $P$ into $E$ is a linear map $D: P \to E$ such that (1) holds. $P$ possesses inner derivations of form [2]

$$D(f) = \{A, f\} \quad \text{some } A \in P \quad \text{and all } f \in P.$$ 

A derivation of $P$ which is not of this form will be called outer.

In this paper we find all the derivations of $P$. We first exhibit (§2) a class of outer derivations of $P$, and in §3 we prove that every derivation of $P$ is a linear combination of one of these and an inner derivation, although this decomposition may not be unique. In particular, we show that a derivation of $P$ which maps constants to zero must be inner. Analogous results hold for derivations of $P$ into $E$. Finally in §4 we characterise certain solutions of the Dirac problem.

2. A class of outer derivations of $P$. It is not obvious at the outset that $P$ does in fact possess outer derivations, but we have

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Lemma 1. For any set of complex numbers \( a = (a_1, \cdots, a_n) \), the map

\[
D_a : f \rightarrow f - \sum_{j=1}^{n} a_j \frac{\partial f}{\partial q_j} + (1 - a_j)q_j \frac{\partial f}{\partial p_j}
\]

is an outer derivation of \( P \).

Proof. Condition (1) can be verified directly, although this is tedious. A more interesting proof is as follows. For each \( f \in P \), let \( X_f \) be the vector field on \( \mathbb{R}^{2n} \) defined by

\[
X_f = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right).
\]

Then for all \( f, g \in P \) we have

\[
[X_f, X_g] = X_f X_g - X_g X_f = -X_{\{f, g\}}, \quad X_f g = \{g, f\} = -\{f, g\}.
\]

Let \( \Omega \) denote the differential 2-form \( \sum_{j=1}^{n} dp_j \wedge dq_j \), defined by

\[
\Omega(X, Y) = \sum_{j=1}^{n} ((X p_j)(Y q_j) - (Y p_j)(X q_j))
\]

for all vector fields \( X, Y \). Let \( \omega \) be any differential 1-form such that \( d\omega = \Omega \). This property is expressed by the equation

\[
\Omega(X, Y) = X \omega(Y) - Y \omega(X) - \omega([X, Y]) \quad \text{all } X, Y.
\]

Now let \( X = X_f, Y = X_g \). After some simple rearranging we get

\[
\{f, g\} - \omega(X_{\{f, g\}}) = \{f - \omega(X_f), g\} + \{f, g - \omega(X_g)\} \quad \text{all } f, g
\]

showing that the map \( f \rightarrow f - \omega(X_f) \) is a derivation of \( P \). We obtain \( D_a \) by putting \( \omega = \sum_{j=1}^{n} (a_j p_j dq_j - (1 - a_j)q_j dp_j) \). Also, since \( D_a(1) \neq 0 \), we see that \( D_a \) is not inner.

Lemma 2. The smallest Lie subalgebra of \( P \) containing the collection of polynomials\(^1\) \( \pi = \{q_i, q_i^2, q_i q_j, q_i^2 q_j, p_i^2\} \) is \( P \) itself.

Proof. Let \( Q \) be the smallest such subalgebra. It is not hard to see that, for each \( i \), \( Q \) contains all the functions of \( q_i \) and \( p_i \) only. From the equations

\[
\{q_i q_{r+1}, q_i p_r^2\} = 2q_i q_{r+1} p_r, \quad \{q_1 q_2 \cdots q_r, q_1 q_{r+1} p_r\} = q_1 q_2 \cdots q_r q_{r+1}
\]

\(^1\) Here and in the sequel \( i \) and \( j \) take the values 1, 2, \cdots, \( n \).
and a simple induction argument, we find that \( q_1 q_2 \cdots q_n \in Q \). Each 
\( f \in P \) of form \( f_1 f_2 \cdots f_n \), where each \( f_i \) is a function of \( q_i \) and \( p_i \) only, 
can be expressed as

\[
f = \{ \cdots \{ \{ q_1 q_2 \cdots q_n, g_1 \}, g_2 \} \cdots, g_n \}\]

where each \( g_i \) is a function of \( q_i \) and \( p_i \) only, such that \( \frac{\partial g_i}{\partial p_i} = f_i \). 
Thus \( Q \) contains all polynomials of product form. Taking linear combinations, we obtain \( Q = P \).

3. **All the derivations of** \( P \). The inner derivations of \( P \) can be 
characterised simply by

**Lemma 3.** If \( D \) is a derivation of \( P \), and \( D(1) = 0 \), then \( D \) is inner.

**Proof.** From (1) we have

\[
\{ D(q_i), p_j \} + \{ q_i, D(p_j) \} = D(\{ q_i, p_j \}) = 0
\]

hence,

\[
\frac{\partial}{\partial q_j} D(q_i) = -\frac{\partial}{\partial p_i} D(p_j).
\]

Similarly,

\[
\frac{\partial}{\partial p_j} D(q_i) = \frac{\partial}{\partial p_i} D(q_j), \quad \frac{\partial}{\partial q_j} D(p_i) = \frac{\partial}{\partial q_i} D(p_j).
\]

Since \( \mathbb{R}^{2n} \) is simply connected, these conditions imply that there 
exists a function \( A \) such that

\[
D(q_i) = -\frac{\partial A}{\partial p_i}, \quad D(p_j) = \frac{\partial A}{\partial q_j}
\]

and \( A \in P \) since the \( D(q_i), D(p_j) \) are all polynomials. The map

\[
S: f \rightarrow D(f) - \{ A, f \} \quad \text{all} \ f \in P
\]

is also a derivation of \( P \) and has the property

\[
(3) \quad S(1) = S(q_i) = S(p_j) = 0;
\]

hence, from (1)

\[
(4) \quad S(\frac{\partial g}{\partial p_i}) = S(\{ q_i, g \}) = \{ q_i, S(g) \} = \frac{\partial}{\partial p_i} S(g) \quad \text{all} \ g \in P.
\]

Similarly,
It is clear from (3), (4), (5) that $S(q_i^2) = \text{constant}$, hence from (1) and (4) we have

$$2S \left( q_i \frac{\partial g}{\partial p_i} \right) = S(\{ q_i, g \}) = \{ S(q_i), g \} + \{ q_i, S(g) \}
$$

$$= 2q_i \frac{\partial}{\partial p_i} S(g) = 2q_i S \left( \frac{\partial g}{\partial p_i} \right) \quad \text{all } g \in P.$$

Substituting $g = q_i p_i, q_j p_i, q_j q_i$, respectively, into (6) gives

$$S(q_i^2) = S(q_i) = S(q_i q_j) = 0.$$

Similarly, we can show that $S(p_i^2) = 0$. Now from (1) we see that $S(f) = S(g) = 0$ implies that $S(\{ f, g \}) = 0$. Since $S(f + g)$ is then also zero, this tells us that $S(f) = 0$ for all $f$ belonging to some Lie subalgebra of $P$ containing $\pi$, that is, for all $f \in P$. The lemma follows immediately from the definition of $S$.

**Lemma 4.** If $D$ is a derivation of $P$ into $E$, and $D(1) = 0$, then $D$ has the form $D(f) = \{ A, f \}$ for some $A \in E$ and all $f \in P$.

**Proof.** Identical to that of Lemma 3 except that the $D(q_i), D(p_i)$ are not now necessarily polynomials.

**Theorem 1.** Every derivation $D$ of $P$ is a linear combination of a derivation of type (2) and an inner derivation. That is, we can find a polynomial $A$ and complex numbers $c, a_i$ such that $D(f) = c D_a(f) + \{ A, f \}$ all $f \in P$.

**Remark.** This decomposition is not necessarily unique, for if $D_a$ and $D_b$ are any two derivations of type (2), we have

$$D_a(f) - D_b(f) = \sum_{i=1}^{n} (b_i - a_i) q_i p_i, f \quad \text{all } f \in P$$

so in fact, we can replace the $D_a$ in the decomposition by any other $D_b$ merely by redefining $A$. In case $D$ is inner, the decomposition is unique modulo additive constants in $A$.

**Proof.** From (1) we find that $D(1) = \text{constant} = c$, say. The map

$$T: f \rightarrow D(f) - c \left( f - \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} \right) \quad \text{all } f \in P$$
is also a derivation of \( P \), and \( T(1) = 0 \). Hence, by Lemma 3, \( T \) is inner. Thus for some constant \( c \) and some \( A \in P \),

\[
D(f) = c \left( f - \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} \right) + \{ A, f \} \quad \text{all } f \in P.
\]

**Theorem 2.** Every derivation \( D \) of \( P \) into \( E \) has the form (7) for some \( A \in E \) and some constant \( c \).

**Proof.** Exactly as for Theorem 1, but using Lemma 4 instead.

These results can be extended to the case of derivations of \( E \) (defined in an obvious manner) provided we topologise \( E \) suitably so that \( P \) forms a dense subset, and make certain continuity assumptions about the derivations. It is obvious from the proofs that Theorems 1 and 2 can be extended to the Lie algebra of real polynomials under Poisson bracket.


A **Dirac map** will mean a solution of the Dirac problem, [5] [6], that is, a linear map \( L \) from \( P \) into the operators acting on some Hilbert space \( H \), such that \( L(1) \) is the identity operator and

\[
[L(f), L(g)] = iL(\{ f, g \}) \quad \text{all } f, g \in P.
\]

Take \( H = L^2(\mathbb{R}^{2n}) \), and let \( K \) denote the \( C^\infty \) complex valued functions on \( \mathbb{R}^{2n} \) with compact support. The condition that the map \( f \mapsto L(f) \) defined by

\[
L(f)\phi = i\{ f, \phi \} + M(f)\phi \quad \text{all } \phi \in K
\]

be a Dirac map, where \( M(f) \) is some \( C^\infty \) function, is

\[
\{ M(f), g \} + \{ f, M(g) \} = M(\{ f, g \}) \quad \text{all } f, g \in P
\]

and

\[
M(1) = 1.
\]

Thus \( f \mapsto M(f) \) must be a derivation of \( P \) into \( E \). Using Theorem 2 and condition (9) we obtain

**Theorem 3.** Every Dirac map of type (8) is of the form

\[
L(f)\phi = i\{ f, \phi \} + \left( f - \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} + \{ A, f \} \right) \phi
\]

for some \( A \in E \).

Souriau’s solution [5] corresponds to \( A = 0 \), and was first found by...
van Hove [7]. Streater [6] obtains the solution corresponding to writing \( A = B - \frac{1}{2} \sum_{j=1}^{n} \rho_{ij} q_j \) in (10). There still remains the question of finding the general Dirac map. Certain other types [4] also reduce to the form (10).

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References


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