A SHORT PROOF OF A THEOREM OF L. JANOS

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Theorem (Janos). Let $X$ be a compact metrizable topological space and $f: X \rightarrow X$ a continuous one-to-one mapping with $\bigcap f^n [X]$ a singleton. Given $\lambda$, $0<\lambda<1$, there exists a metric $\rho$ on $X$ such that the metric topology of $(X, \rho)$ is identical with the original one and $\rho(f(x), f(y)) = \lambda \rho(x, y)$ for all $x, y \in X$.

It is the purpose of this note to provide a proof of a somewhat stronger result, which seems to be shorter and simpler than any of those outlined by Janos in [1], [2] and [3].

Theorem. Let $X$ be a compact metrizable topological space and $f: X \rightarrow X$ a continuous one-to-one mapping with $\bigcap f^n [X] = \{x_0\}$, where $x_0 \in X$. Given $\lambda$, $0<\lambda<1$, a homeomorphism $h$ of $X$ into $l_2$ exists such that

$$\|h(f(x')) - h(f(x''))\| = \lambda \|h(x') - h(x'')\| \quad \text{for all } x', x'' \in X.$$

Proof. We may assume that $X \sim \{x_0\} \neq \emptyset$. Let $B$ be a countable base for (the open nonempty set) $X \sim f[X]$. To each pair $U, V \in B$, such that $U \subset V$, we make correspond a continuous $\phi: X \rightarrow [0, 1]$ such that $\phi[U] = 1$ and $\phi[X \sim V] = 0$. Using the odd positive integers as an index set we obtain the family $\{\phi_{2n-1}: n = 1, 2, \ldots \}$ of mappings. Since $f[X]$ is closed and $\phi_{2n-1} f^{-1}: f[X] \rightarrow [0, 1]$ is continuous for each $n = 1, 2, \ldots$ there exists, by the Tietze extension theorem, a continuous $\phi_{2n-1} f^{-1}: f[X] \rightarrow [0, 1]$ is continuous for each $n = 1, 2, \ldots$ there exists, by the Tietze extension theorem, a continuous $\phi_{2n-1} f^{-1}: f[X] \rightarrow [0, 1]$ which coincides with $\phi_{2n-1} f^{-1}$ on $f[X]$. Thus $\phi_{2n-1}(f(x)) = \phi_{2n-1}(x)$ ($n = 1, 2, \ldots ; x \in X$). Assuming $\phi_{2n-1}(2n-1): X \rightarrow [0, 1]$ defined, continuous and satisfying

$$\phi_{2n-1}(2n-1)(f(x)) = \phi_{2n-1}(2n-1)(x) \quad (n = 1, 2, \ldots ; x \in X),$$

we define $\phi_{2n}(2n-1): X \rightarrow [0, 1]$ by choosing a continuous extension of $\phi_{2n-1}(2n-1): f[X] \rightarrow [0, 1]$ to the whole of $X$, thereby obtaining continuous mappings $\phi_{2n}(2n-1)$ of $X$ into $[0, 1]$ for all $m = 0, 1, \ldots$, $n = 1, 2, \ldots$ and satisfying

$$\phi_{2n}(2n-1)(f(x)) = \phi_{2n-1}(2n-1)(x) \quad (m = 1, 2, \ldots ; n = 1, 2, \ldots, x \in X).$$

Also, clearly,

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\[ \phi_{(2n-1)}(f(x)) = 0 \quad (n = 1, 2, \ldots ; x \in X). \]

We now define \( h \) as follows. If \( k = 2^m(2n-1) \) set
\[ y_k = \lambda^{m+n} \phi_{2^m(2n-1)}(x) \quad \text{and} \quad h(x) = (y_1, y_2, \ldots, y_k, \ldots). \]

Obviously \( h(x) \in \mathcal{L}_2 \). It is a straightforward matter to verify that \( h \) is one-to-one and continuous; hence, by compactness of \( X \), a homeomorphism onto \( h[X] \). Finally,
\[
\| h(f(x')) - h(f(x'')) \|_2^2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)}[\phi_{2^m(2n-1)}(f(x')) - \phi_{2^m(2n-1)}(f(x''))]_2^2
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)}[\phi_{2^{m-1}(2n-1)}(x') - \phi_{2^{m-1}(2n-1)}(x'')]_2^2
\]
\[
= \lambda^{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)}[\phi_{2^m(2n-1)}(x') - \phi_{2^m(2n-1)}(x'')]_2^2
\]
\[
= \lambda^{2} \| h(x') - h(x'') \|_2^2
\]
and the theorem follows.

REFERENCES


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