PERMUTATIONS AND TWO SEQUENCES WITH THE 
SAME CLUSTER SET

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Suppose that a (n→a_n) and b (n→b_n) are sequences in a compact
metric space with distance d. Suppose further that a and b have
the same set of cluster points C. Von Neumann [1, p. 11–12] proved
there exists a permutation \( \pi \) of the set of positive integers \( Z \) such that
\( d(a_n, b_{\pi n}) \to 0 \). Halmos [2] recently gave an improved proof (and the
above statement for compact metric spaces). A discussion of this
result may be found in [2]. My purpose is to give a shorter proof than
that of Halmos, which used the Schröder-Bernstein Theorem.

Proof. Let \( \pi 1 = 1 \). We now construct \( \pi n \) inductively for \( n > 1 \), given
\( \pi 1, \ldots, \pi (n-1) \). Write \( Z_1 \) for \( Z \) and \( Z_n \) for \( Z - \{ \pi 1, \ldots, \pi (n-1) \} \).
Let \( \rho (n) = \min Z_n \). Let \( \pi n \) be the smallest integer in \( Z_n \) such that
\[ d(b_{\pi n}, a_n) \leq d(a_n, C) + 1/\rho (n) + d(b_{\rho (n)}, C). \]

Such an element exists: Let \( c_n \) be chosen in the compact set \( C \) so that
\( d(a_n, c_n) = d(a_n, C) \). Then for some \( m \in Z_n \), \( d(b_m, c_n) \leq 1/\rho (n) \), since \( c_n \)
is a cluster point of \( b \). Then \( \pi n = m \) would satisfy (1). To see that \( \pi \)
is “onto,” i.e., \( \{ \pi 1, \pi 2, \ldots \} = Z \), let us suppose not. Let \( q = \min Z - \{ \pi 1, \pi 2, \ldots \} \). Then for all \( n \), \( \rho (n) \leq q \), and \( N \)
\[ \rho (n) = \max \{ n : q \leq \rho (n) \} \] is finite. Choose \( c^* \in C \) so that
\( d(b_{q}, c^*) = d(b_{q}, C) \). Since \( c^* \in C \), \( c^* \) is a cluster point of \( a \), so we may choose \( n > N \) such that
\( 1/q > d(a_n, c^*) \). Obviously, \( d(a_n, C) + 1/q > d(a_n, c^*) \). Since \( n > N \), \( \rho (n) = q \), and since \( \pi n > q \), \( d(b_{q}, a_n) > d(a_n, C) + 1/q + d(b_{q}, c^*) > d(a_n, c^*) + d(b_{q}, c^*) \), contradicting the triangle inequality. Hence \( \pi \) is onto.

Since \( \pi n \to \infty \) and \( d(a_n, C) \to 0 \) and \( d(b_{\pi n}, C) \to 0 \), (1) implies \( d(a_n, b_{\pi n}) \to 0 \).

References

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