THE STONE-ČECH COMPACTIFICATION OF AN IRREDUCIBLY CONNECTED SPACE

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1. Introduction. A connected topological space $X$ is irreducibly connected about a subset $A \subseteq X$ (written $X$ is an $A$-i-connex) if no proper connected subspace of $X$ contains $A$. A connected space $Y$ is irreducibly closed connected about a subset $B \subseteq Y$ (written $Y$ is a $B$-i-C-connex) if no proper closed connected subspace of $Y$ contains $B$. The structure of such spaces has been studied by Gehman [1], Wilder [7] and Strebe [3], [4], among others. In this paper we show that the Stone-Čech compactification $\beta X$ of an $i$-connex $X$ is an $i$-connex, and that $\beta Y$, for $Y$ a suitably restricted $i$-C-connex, is an $i$-C-connex.

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2. Notation. All spaces are at least $T_1$ and completely regular. For $S \subseteq X$, $\text{Cl}_S$ is the closure of $S$ in $X$, and for $T \subseteq \beta X$, $\text{Cl}_T$ $T$ is the closure of $T$ in $\beta X$. If $S \subseteq X$, then $S^0 = \beta X - \text{Cl}_S (X - S)$.

3. i-connexes.

Theorem 1. If $X$ is an $A$-i-connex, then $\beta X$ is an $A \cup (\beta X - X)$-i-connex.

Proof. Since $X$ is connected, each point of $\beta X - X$ is a noncut point of $\beta X$. Thus any set about which $\beta X$ is an $i$-connex must contain $\beta X - X$ [1, Theorem 3, p. 545].

If there is a subspace $X' \neq \beta X$ with $A \cup (\beta X - X) \subseteq X'$, then for any $x \in \beta X - X'$, $x \in X - A$ so $x$ is a cut point of $X$. However, any cut point of $X$ is a cut point of $\beta X$ because an open subset $U$ of $\beta X$ is connected if and only if $UT \cap X$ is connected [2, Lemma 1.4, p. 575]. Thus for $x \in \beta X - X'$, $\beta X - \{x\} = P \cup Q$ (sep).

If $X'$ is connected, then $X' \subseteq P$, say. Now $Q \cup \{x\}$, being a continuum, has a noncut point $z \neq x$ [5, Theorem 1.11, p. 491]. Further $z \in X - A$. It follows that

$$\beta X - \{z\} = (P \cup \{x\}) \cup [(Q \cup \{x\}) - \{z\}]$$

is connected; hence that $X - \{z\}$ is a connected proper subspace of $X$ containing $A$, a contradiction.

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To see that no converse is possible, let \( X \) be the nonnegative reals with the usual topology and \( A = \{0\} \).

4. \( i \)-\( C \)-connexes.

**Theorem 2.** If the \( A \)-\( i \)-\( C \)-connex \( X \) is normal and is semilocally connected at each \( x \in X - A \), then \( \beta X \) is an \( A \)-\( i \)-\( C \)-connex.

**Proof.** If there is a proper closed connected subspace \( K \) of \( \beta X \) with \( A \subset K \), then for \( x \in X \cap (\beta X - K) \) there is an \( X \)-open neighborhood \( V \) of \( x \) with \( \text{Cl}_\beta V \cap K = \emptyset \). Since \( X \) is semilocally connected at \( x \), there is an \( X \)-open set \( U \) with \( x \in U \subset V \) such that \( X - U \) has finitely many components \( C_1, \ldots, C_n \). Further, \( n \geq 2 \) and \( A \cap C_i \neq \emptyset \) for at least two indices \( i \).

Now \( X \) is normal and therefore the closures in \( \beta X \) of \( C_1, \ldots, C_n \) are mutually disjoint closed connected sets [6, Theorem 1, p. 97], i.e., they are the components of \( \text{Cl}_\beta(X - U) \). It follows that \( A \subset K \subset \text{Cl}_\beta C_j \) for some index \( j \), a contradiction.

**Theorem 3.** If the \( A \)-\( i \)-\( C \)-connex \( X \) is normal and locally connected, then \( \beta X \) is an \( A \)-\( i \)-\( C \)-connex.

**Proof.** If there is a proper closed connected subspace \( K \) of \( \beta X \) with \( A \subset K \), then there is a connected \( X \)-open set \( U \) with \( \text{Cl}_\beta U \cap K = \emptyset \). \( X - U \) is not connected, so from \( X - U = \text{Cl}(X - \text{Cl} U) \) it follows that \( X - \text{Cl} U \) is not connected.

If \( C \) is a component of \( X - \text{Cl} U \), then \( C \) is open in \( X \) and \( X - C \) is closed and connected. Therefore \( C \cap A \neq \emptyset \) for each component \( C \) of \( X - \text{Cl} U \). Thus \( X - \text{Cl} U = P \cup Q \) (sep) with \( P \cap A \neq \emptyset \), \( Q \cap A \neq \emptyset \).

Now \( K \subset \beta X - \text{Cl}_\beta U = (X - \text{Cl} U)^0 = (P \cup Q)^0 \), and by [6, Lemma 2, p. 98] \((P \cup Q)^0 = P^0 \cup Q^0 \) and \((P \cap Q)^0 = P^0 \cap Q^0 \). Since \( K \) is connected, \( K \subset P^0 \), say; however, \( A \cap Q^0 \supset A \cap Q \neq \emptyset \) and this is a contradiction.

**Bibliography**


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