A NEW PROOF THAT METRIC SPACES ARE PARACOMPACT

MARY ELLEN RUDIN

By using a well-ordered open cover, there is a simple proof of the nice theorem [1] that every metric space is paracompact.

Assume that X is a metric space and that $\{C_\alpha\}$ is an open cover of X indexed by ordinals. Let ρ be a metric on X and let $S(x, r)$ be the open sphere with center x and radius r. For each positive integer n define $D_{\alpha n}$ (by induction on n) to be the union of all spheres $S(x, 2^{-n})$ such that:

(1) α is the smallest ordinal with $x \in C_\alpha$,

(2) $x \notin D_{\beta j}$ if $j < n$,

(3) $S(x, 3\cdot 2^{-n}) \subset C_\alpha$.

Then $\{D_{\alpha n}\}$ is a locally finite refinement of $\{C_\alpha\}$ which covers X; hence X is paracompact.

Certainly $\{D_{\alpha n}\}$ refines $\{C_\alpha\}$. To see that $\{D_{\alpha n}\}$ covers X, observe that, for $x \in X$, there is a smallest ordinal α such that $x \in C_\alpha$, and an n so large that (3) holds. Then, by (2), $x \in D_{\beta j}$ for some $j \leqq n$.

To prove that $\{D_{\alpha n}\}$ is locally finite, assume an $x \in X$ and let α be the smallest ordinal such that $x \in D_{\alpha n}$ for some n, and choose j so that $S(x, 2^{-j}) \subset D_{\alpha n}$. The proof consists of showing that:

(a) if $i \geqq n+j$, $S(x, 2^{-n-j})$ intersects no $D_{\beta i}$,

(b) if $i < n+j$, $S(x, 2^{-n-j})$ intersects $D_{\beta i}$ for at most one β.

Proof of (a). Since $i > n$, by (2), every one of the spheres of radius 2^{-i} used in the definition of $D_{\beta i}$ has its center y outside of $D_{\alpha n}$. And since $S(x, 2^{-j}) \subset D_{\alpha n}$, $\rho(x, y) \geqq 2^{-j}$. But $i \geqq j+1$ and $n+j \geqq j+1$, so $S(x, 2^{-n-j}) \cap S(y, 2^{-i}) = \emptyset$.

Proof of (b). Suppose $p \in D_{\beta i}$, $q \in D_{\gamma i}$, and $\beta < \gamma$; we want to show that $\rho(p, q) > 2^{-n-j+1}$. There are points y and z such that $p \in S(y, 2^{-i}) \subset D_{\beta i}$, $q \in S(z, 2^{-i}) \subset D_{\gamma i}$; and, by (3), $S(y, 3\cdot 2^{-i}) \subset C_\beta$ but, by (2), $z \notin C_\beta$. So $\rho(y, z) \geqq 3\cdot 2^{-i}$ and $\rho(p, q) > 2^{-i} \geqq 2^{-n-j+1}$.

Bibliography

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University of Wisconsin

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