EQUICONTINUITY AND $n$-LENGTH

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Let $(M, \rho)$ be a pseudo-metric space. We shall obtain a necessary and sufficient condition that a collection of curves can be parametrized in such a manner that the collection of parametrizations be equicontinuous. This result can be extended to the case where $\rho$ is a quasi-pseudo-metric. The $\mu$-length defined here differs essentially from that originally defined by M. Morse in $[M]$. We also use ideas of $[F]$ and $[S]$.

Let $I = [a, b]$. If $u \in I$, then let $I_u = [a, u]$. If $(N, \sigma)$ is a pseudo-metric space and if $f: I \to N$, then let $\omega(f; J) = \sup \{ \sigma(f(u), f(v)) | u, v \in J \}$ whenever $J$ is an interval contained in $I$. Let $C(I)$ be the space of continuous functions on $I$ into $(M, \rho)$ metrized by $\sigma$ where $\sigma(x, y) = \sup \{ \rho(x(u), y(u)) | u \in I \}$.

For each positive integer $n$ and $x \in C(I)$, let

$$\mu_n x = \sup \left\{ \sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) | a \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq b \right\}.$$ 

Since $x$ is continuous and $I$ is compact, the supremum is a maximum. Evidently $\mu_n \leq \mu_{n+1}$, $\mu_n \leq n \mu_1$ and $|\mu_n x - \mu_n y| \leq 2n \sigma(x, y)$.

**Lemma 1.** Let $A \subset C(I)$ be equicontinuous and $M_n = \sup \{ \mu_n x | x \in A \}$. The $\lim_{n \to \infty} n^{-1} M_n = 0$.

**Proof.** Let $\epsilon > 0$. There exists $\delta > 0$ such that $\omega(x; J) < \epsilon$ whenever length $J = |J| < \delta$. Let $K > \delta^{-1} |I|$ and let $T_j, j = 1, 2, \cdots, K - 1$, be points in $I$ which divide $I$ into $K$ equal subintervals. Let $t_i \in I$, $i = 0, 1, \cdots, n$, and let $\{\sigma_m\}, m = 0, 1, \cdots, R = n + K - 1$, be a non-decreasing arrangement of $\{t_i\} \cup \{T_j\}$. If $x \in A$ then

$$\sum_{i=1}^{n} \rho(x(t_{i-1}), x(t_i)) \leq \sum_{m=1}^{R} \rho(x(\sigma_{m-1}), x(\sigma_m)) < (n + K - 1) \epsilon.$$

If $x \in C(I)$ and $u \in I$ let $\phi_{n,x}(u) = \mu_n x | I_u$.

**Lemma 2.** If $n > 2$, then $\phi_{n-2,x} \leq (1 - 3/n) \phi_{n,x}$.

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Proof. It is sufficient to show that $\mu_{n-3}x \geq (1-3/n)\mu_n x$. There exists a nondecreasing sequence $\{t_i\}$ in $I$ such that $\sum_{i=1}^n a_i = \mu_n x$, where $a_i = \rho(x(t_{i-1}), x(t_i))$. Let

$$b_i^j = a_i \quad \text{if } i > j + 1 \lor i < j - 1,$$

$$= 0 \quad \text{if } j - 1 \leq i \leq j + 1$$

for $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, n$.

If $A = \sum_{j=1}^n \sum_{i=1}^n b_i^j$, then

$$A = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} a_i + \sum_{i=1}^{n-2} \sum_{j=0}^{i-2} a_i$$

$$= \sum_{i=1}^n \sum_{j=1}^i a_i + \sum_{i=1}^n \sum_{j=i+1}^n a_i$$

$$= \sum_{i=1}^n (i - 2) a_i + \sum_{i=1}^n (n - i - 1) a_i$$

$$= (n - 3) \sum_{i=1}^n a_i + a_1 + a_n.$$ 

Hence there exists $k$ such that $\sum_{i=1}^n b_i^k \geq (1-3/n)\mu_n x$. If $t_{k-1}$ and $t_k$ are deleted from $\{t_i\}$ and if the resulting nondecreasing sequence is labelled $\{\sigma_j\}, j = 0, 1, \ldots, n-2$, then

$$\mu_{n-3}x \geq \sum_{j=1}^{n-2} \rho(x(\sigma_{j-1}), x(\sigma_j)) \geq \sum_{i=1}^n b_i^k \geq (1-3/n)\mu_n x.$$ 

We now define a $\mu$-length on $C(I)$ by $\mu = \sum_{j=1}^{n-1} 2^{-j} \mu_n$. Thus $\mu_1 \leq \mu \leq \mu_1 \sum_{j=1}^{n-1} 2^{-j} n = 2\mu_1$. If $x \in C(I)$, let $\phi_x(u) = \mu x | I_u$. Evidently $2^{-k}\omega(\phi_x, J) \leq \omega(\phi_x; J) \leq 2\omega(x; J)$.

**Lemma 3.** $\omega(x, J) \leq \max \{6n^{-1}\mu_n x, 2^{n+1}\omega(\phi_x; J)\}$ for all $n$.

**Proof.** Let $u, v \in J$ with $u < v$ and let $2\eta = \rho(x(u), x(v))$. If $\eta > 3n^{-1}\mu_n x$, then

$$\phi_{x,v}(v) - \phi_{x,u}(u) \geq \phi_{x-2,x}(u) + 2\eta - \phi_{x,x}(u) \geq 2\eta - 3n^{-1}\phi_{x,x}(u) > \eta.$$ 

Hence $\eta \omega(\phi_{x,x}; J) \leq 2^n \omega(\phi_x; J)$.

**Corollary.** A necessary and sufficient condition that $\phi_x$ be constant on $J$ is that $x$ be constant on $J$.

Let $C = \{X \in C([0, 1])| \phi_x(u) = (\mu X)u \text{ for all } u \in [0, 1]\}$. Evidently $\omega(\phi_x; J) = (\mu X)| J|$ and
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\[
2^{-1}(\mu X) | J | \leq \omega(X; J) \leq \max \{ 6n^{-1}(\mu_n X), 2^{n+2}(\mu_n X) | J | \}
\]

for all \( n \) whenever \( X \in C_\mu \).

**Lemma 4.** Let \( A \subset C_\mu \) and \( M_n = \sup \{ \mu_n X \mid X \in A \} \). Then \( A \) is equicontinuous if and only if \( \lim_{n \to \infty} n^{-1}M_n = 0 \).

**Proof.** Choose \( \varepsilon > 0 \) and take \( j \) so that \( 6M_j < j \varepsilon \). Let \( \delta = \left[ 2^{j+2}M_1 \right]^{-1} \varepsilon \).

If \( u \leq v < u + \delta \), then

\[
\rho(X(u), X(v)) \leq \omega(X; [u, v]) \leq \max \{ 6j^{-1}M_j, 2^{j+2}M_1 \delta \} = \varepsilon
\]

and the proof is complete because of Lemma 1.

**Lemma 5.** Let \( X_n \subset C_\mu \). Then \( \lim_{n \to \infty} n^{-1}(\mu_n X_n) = 0 \) if and only if \( \{ X_n \} \) is equicontinuous.

**Proof.** If \( \{ X_n \} \) is equicontinuous, then

\[
0 = \lim_{k \to \infty} k^{-1} \sup \mu_k X_n \geq \lim_{k \to \infty} k^{-1}(\mu_k X_k) \geq 0.
\]

The proof in the other direction is like that of the preceding lemma.

If \( x \in C(I) \) and \( y \in C(J) \), let \( D_F(x, y) = \inf \{ \sigma(x, y \circ h) \mid h \text{ is a sense-preserving homeomorphism of } I \text{ onto } J \} \). Then \( D_F \) is a pseudo-metric and is a metric on the space of Fréchet equivalence classes: \( x \sim y \) if \( D_F(x, y) = 0 \). Such an equivalence class is a Fréchet curve. If \( m \) is continuous and monotonically nondecreasing from \( I \) onto \( J \) then \( z \in F \) where \( z \) is defined by \( x = z \circ m \). It is easy to see that \( (\mu_n x - \mu_n y) \leq 2nD_F(x, y) \) and \( |\mu_n - \mu_x| \leq 4D_F(x, y) \).

If \( \xi \) and \( \eta \) are Fréchet curves with \( x \in \xi \) and \( y \in \eta \), then let \( \mu_\xi = \mu_n x, \mu_\zeta = \mu_n \) and \( D_F(\xi, \eta) = D_F(x, y) \).

**Lemma 6.** If \( X, Y \subset C_\mu \) and \( X \sim Y \), then \( X = Y \).

**Proof.** There exist homeomorphisms \( \{ h_k \} \) such that \( \sigma(X, Y \circ h_k) < k^{-1} \). If \( u \in [0, 1] \) then \( (\mu X) | u - h_k(u) | = | \phi_x(u) - \phi_y(h_k(u)) | < k^{-1} \)

so that \( h_k \) converges uniformly to the identity on \( [0, 1] \).

If \( x \in C(I) \), let \( X = x \circ \phi x / \mu x \) if \( x \) is not constant, and let \( X = x(a) \) otherwise. Thus, by some earlier remarks, \( X \sim F x \).

**Lemma 7.** If \( x \in C(I) \), then \( X \subset C_\mu \).

**Proof.** If \( u \in [0, 1] \), there exists \( c \in I \) such that \( (\mu x)u = \phi_x(c) \). Thus \( (X | [0, u]) F (x | I_c) \) so that \( \phi_x(u) = \phi_x(c) = (\mu x)u = (\mu X)u \).

It follows that each Fréchet curve has a unique representation in \( C_\mu \).

Let \( F \) be the space of Fréchet curves metrized by \( D_F \). If \( x, y \in \xi \subset F \) then range \( x = \text{range } y \). Let \( [\xi] \) denote range \( x \).
THEOREM. Let $K$ be a compact subset of $M$ and $B \subseteq F$. If $[\xi] \subseteq K$ for all $\xi \in B$ then $B$ is sequentially compact if and only if $\lim_{n \to \infty} n^{-1}M_n = 0$ where $M_n = \sup \{\mu_\xi \mid \xi \in B\}$. If $[\xi_n] \subseteq K$, then $\{\xi_n\}$ contains a convergent subsequence if and only if $\lim_{n \to \infty} n^{-1}(\mu_{\xi_n}) = 0$.

REFERENCES


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