1. This paper is a study of asymptotic properties of solutions of the differential equation

$$(L) \quad y''' + p(t)y' + q(t)y = 0.$$  

Throughout, we shall assume that $p(t)$, $p'(t)$ and $q(t)$ are continuous, and $p(t)$, $q(t)$ are bounded and do not change sign on $[a, \infty)$, $a \geq 0$. Two theorems are provided here, and the techniques used are similar to ones used by Lazer [3], Švec [4] and Zlámal [5] in previous studies of this differential equation. The proofs are based on the following three lemmas. The first lemma is the result due to E. Esclangon [2] (for another source see [1]) and the other two lemmas are elementary and will not be proved here.

**Lemma 1.1.** Let the function $p_i(t)$, $i = 0, 1, \cdots, n$ be continuous and bounded for $t \geq t_0$. If

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = p_0(t)$$

and $y(t)$ is bounded for $t \geq t_0$, then its derivatives $y^{(k)}(t)$ ($1 \leq k \leq n$) are also bounded for $t \geq t_0$.

**Lemma 1.2.** Let $f(t) \in C^1[a, \infty)$. If $\int_a^\infty f^2(t)dt < \infty$ and $f'(t)$ is bounded, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

**Lemma 1.3.** Let $f(t) \in C^2[a, \infty)$. If $f(t) \rightarrow 0$ as $t \rightarrow \infty$ and $f''(t)$ is bounded, then $f'(t) \rightarrow 0$ as $t \rightarrow \infty$.

I am indebted to Professor A. C. Lazer for many fruitful conversations concerning this differential equation.

2. In this section we consider the behavior of solutions of $(L)$ subject to the conditions $p(t) \leq 0$ and $p'(t) - 2q(t) \geq A > 0$. We will use several times the following identity, which has played an important role in most of the previous investigations of $(L)$. If $y(t)$ is a solution of $(L)$ and

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\[ F(y(t)) = y''(t) - 2y(t)y'''(t) - p(t)y^2(t), \]
then
\[ F(y(t)) = F(y(a)) - \int_a^t (p'(t) - 2q(t))y^2(t)dt. \]

From (1) it follows that if \( y(t) \) is a nontrivial solution of (L) and \( p'(t) - 2q(t) \geq A > 0 \), then \( F(y(t)) \) is strictly decreasing.

**Lemma 2.1.** Let \( p(t) \leq 0, p'(t) - 2q(t) \geq A > 0 \). If \( y(t) \) is a solution of (L) for which \( F(y(t)) > 0 \) for all \( t \in [a, \infty) \), then \( y^{(k)}(t) \rightarrow 0 \) as \( t \rightarrow \infty, \) \( k = 0, 1, 2, 3. \)

**Proof.** Since \( F(y(t)) > 0, p'(t) - 2q(t) \geq A > 0 \), it follows from (1) that for all \( t \geq a, \)
\[ \int_a^t y^2(s)ds \leq \frac{F(y(a))}{A}, \]
and hence
\[ \int_a^\infty y^2(s)ds < \infty. \]

We assert that \( y'(t) \) is bounded. There are two possibilities.

(a) \( y''(t) \) has arbitrarily large zeros.

(b) There exists a number \( c \) such that \( y''(t) \neq 0 \) for \( t \geq c. \)

In case of possibility (a), \( y'(t) \) has arbitrarily large maxima, minima. At every maxima and minima of \( y'(t) \), we have
\[ y^2(t) - p(t)y^2(t) = F(y(a)) - \int_a^t (p'(t) - 2q(t))y^2(t)dt \]
or
\[ y^2(t) \leq F(y(a)) + p(t)y^2(t) - A \int_a^t y^2(t)dt \leq F(y(a)). \]
Thus \( y'(t) \) is bounded at its maxima and minima and hence bounded on \([a, \infty).\)

In case of possibility (b), since \( y''(t) \neq 0 \) for \( t \geq c, \) \( y'(t) \) has constant sign after some \( t, \) say \( t = t_4 \geq c, \) and thus either \( y'(t)y''(t) < 0 \) or \( y'(t)y''(t) > 0 \) for \( t \geq t_4. \) Because \( \int_a^\infty y^2(s)ds \) is convergent, \( y'(t) \) and \( y''(t) \) cannot have the same sign, and thus we have \( y'(t)y''(t) < 0 \) and from this our assertion follows.

From the boundedness of \( y'(t) \) and \( \int_a^\infty y^2(s)ds < \infty \) and Lemma 1.2,
we conclude that \( \lim_{t \to -\infty} y(t) = 0 \). Now from Lemmas 1.1 and 1.3, it follows at once that

\[
\lim_{t \to -\infty} y'(t) = \lim_{t \to -\infty} y''(t) = \lim_{t \to -\infty} y'''(t) = 0.
\]

**Lemma 2.2.** Let \( p(t) \leq 0 \) and \( p'(t) - 2q(t) \geq A > 0 \). If \( z_2(t) \) is the solution of (L) defined by the initial conditions \( z_2(a) = z_2'(a) = 0 \), \( z_2''(a) = 1 \), then \( \lim_{t \to -\infty} z_2(t) = \infty \).

**Proof.** Since \( F[z_2(a)] = 0 \) and \( F[z_2(t)] \) is strictly decreasing, \( F[z_2(t)] = z_2^2(t) - 2z_2(t)z_2''(t) - p(t)z_2'''(t) < 0 \) for \( t > a \). Since \( p(t) \leq 0 \), \( z_2(t) > 0 \), \( z_2'(t) > 0 \) and \( z_2''(t) > 0 \) for \( t > a \) from which the assertion follows.

**Theorem 2.3.** If \( p(t) \leq 0 \) and \( p'(t) - 2q(t) \geq A > 0 \), then there exist two independent nontrivial solutions \( u(t) \) and \( v(t) \) of (L) which tend to zero with their first three derivatives. If \( y(t) \) is any nontrivial solution of (L) which is not a linear combination of \( u(t) \) and \( v(t) \), then \( |y(t)| \) tends to infinity as \( t \) tends to infinity.

**Proof.** Let \( z_0, z_1, z_2 \) be the solutions of (L) satisfying the initial conditions

\[
\begin{align*}
z_i^{(j)} &= \delta_{ij}, & i \neq j, \\
&= 1, & i = j, \quad i, j = 0, 1, 2.
\end{align*}
\]

For each integer \( n > a \), let \( b_{0n}, b_{2n} \) and \( c_{1n}, c_{2n} \) be numbers such that

\[
\begin{align*}
b_{0n}z_0(n) + b_{2n}z_2(n) &= 0, \\
c_{1n}z_1(n) + c_{2n}z_2(n) &= 0
\end{align*}
\]

and

\[
\begin{align*}
b_{0n}^2 + b_{2n}^2 &= c_{1n}^2 + c_{2n}^2 = 1.
\end{align*}
\]

Let \( u_n(t) \) and \( v_n(t) \) be the nontrivial solutions of (L) defined by

\[
\begin{align*}
u_n(t) &= b_{0n}z_0(t) + b_{2n}z_2(t), \\
v_n(t) &= c_{1n}z_1(t) + c_{2n}z_2(t).
\end{align*}
\]

Since \( u_n(n) = v_n(n) = 0 \), we have \( F(u_n(n)) \geq 0 \), \( F(v_n(n)) \geq 0 \). Because \( F(y(t)) \) is a decreasing function, it follows that

\[
\begin{align*}
F(u_n(t)) &> 0, \\
F(v_n(t)) &> 0 \quad \text{for } t \in [a, n).
\end{align*}
\]

Now by (3) there exists a sequence of integers \( \{n_j\} \) such that the sequences \( \{b_{0nj}\}, \{b_{2nj}\} \) and \( \{c_{1nj}\}, \{c_{2nj}\} \) converge respectively to numbers \( b_0, b_2, c_1 \) and \( c_2 \) such that

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Let \( u(t) \) and \( v(t) \) be the solutions of (L) defined by

\[
u(t) = b_0 z_0(t) + b_2 z_2(t), \quad v(t) = c_1 z_1(t) + c_2 z_2(t).\]

By the linear independence of \( z_0, z_1 \) and \( z_2 \) and from (5), it follows that \( u \) and \( v \) are nontrivial solutions of (L). Clearly the sequences \( \{u_k(t)\} \) and \( \{v_k(t)\} \) converge to \( u(t) \) and \( v(t) \), respectively, and from (4) it follows that \( F(u(t)) \geq 0 \) and \( F(v(t)) \geq 0 \) for all \( t \in [a, \infty) \). Hence, by Lemma 2.1, \( u_k(t) \to u(t) \) and \( v_k(t) \to v(t) \) as \( t \to \infty \), \( k=0, 1, 2, 3 \).

If \( u(t) \) and \( v(t) \) were dependent, then from (5) it would follow that \( u(t) = K z_2(t) \), for some \( K \neq 0 \). Since \( z_2(a) = z_2'(a) = 0 \) and \( z_2''(a) = 1 \), we have

\[
F(u(t)) = - \int_a^t (p'(t) - 2q(t)) v^2(t) dt < 0 \quad \text{for } t > a,
\]

which is contradictory. Thus \( u(t) \) and \( v(t) \) are independent.

Consider the solution \( u(t) \), \( v(t) \) and \( z_2(t) \). Now there are two possibilities—that either \( u(t) \), \( v(t) \) and \( z_2(t) \) are dependent or that they are independent. Suppose these are dependent. We can find numbers \( B \), \( C \) and \( D \), not all zero such that

\[
w(t) = Bu(t) + Cv(t) + Dz_2(t) = 0.
\]

By the independence of \( u(t) \) and \( v(t) \), \( D \neq 0 \). By Lemma 2.2, we have

\[
\lim_{t \to \infty} |z_2(t)| = \infty , \quad \text{and thus when } t \to \infty, w(t) \to \infty, \quad \text{which is not possible. Hence } u(t), v(t) \text{ and } z_2(t) \text{ are independent, and since the order of } (L) \text{ is three, it follows from the theory of linear differential equations that every solution } y(t) \text{ of } (L) \text{ is of the form } y(t) = a u(t) + b v(t) + c z_2(t), \text{ for some constants } a, b \text{ and } c. \text{ Since } \lim_{t \to \infty} |z_2(t)| = \infty, y(t) \to 0 \text{ as } t \to \infty \text{ if and only if } c = 0. \text{ From this, assertions of the theorem follow immediately.}
\]

3. We will now consider the behavior of the solutions of (L) subject to the conditions \( p(t) \leq 0 \), and \( 2q(t) - p'(t) \geq d > 0 \). Under these conditions, if \( y(t) \) is any nontrivial solution of (L), then the function \( F(y(t)) \) is strictly increasing.

**Lemma 3.1.** If \( p(t) \leq 0, 2q(t) - p'(t) \geq d > 0 \text{ and } y(t) \text{ is any solution of (L) satisfying the initial conditions}

\[
y(c) = 0, \quad y'(c) = 0 \quad \text{and} \quad y''(c) > 0,
\]

(\text{where } c \text{ is an arbitrary number greater than } a), \text{ then}
\( y(t) > 0, \quad y'(t) < 0, \quad y''(t) > 0 \) for all \( t \in [a, c) \).

**Proof.** Since \( y(c) = 0, \ y'(c) = 0 \) and \( F(y(t)) \) is an increasing function, we have \( F(y(t)) < 0 \) in \( [a, c) \), and thus \( y(t)y''(t) > 0 \) in \( [a, c) \). From this and \( y''(c) > 0 \), it follows that \( y''(t) > 0 \) and \( y(t) > 0 \) in \( [a, c) \). Since \( y'(t) \) is an increasing function in \( [a, c) \) and \( y'(c) = 0 \), we have \( y'(t) < 0 \), \( t \in [a, c) \), which proves the lemma.

**Lemma 3.2.** If \( p(t) \leq 0, \ 2q(t) - p'(t) \geq d > 0 \) and \( y(t) \) be a nontrivial solution of \( (L) \) for which \( F(y(b)) = 0 \) for some \( b \in [a, \infty) \), then \( y(t) \) is unbounded on \( [a, \infty) \).

**Proof.** Suppose \( y(t) \) is a nontrivial bounded solution of \( (L) \) with \( F(y(b)) \neq 0, \ b \geq a \). Since \( p(t), q(t) \) and \( y(t) \) are bounded by Lemma 1.1, \( y'(t) \) and \( y''(t) \) are also bounded, and thus \( F(y(t)) \) is bounded. Hence

\[
\int_a^\infty y^2(s)ds < \infty.
\]

Since \( y'(t) \) is bounded, it follows from the Lemmas 1.2 and 1.3 that

\[
limit_{t \to \infty} y(k)(t) = 0, \ k = 0, 1, 2.
\]

Thus \( \lim_{t \to \infty} F(y(t)) = 0 \). This is a contradiction because \( F(y(b)) \neq 0 \) for \( b \geq a \) and \( F(y(t)) \) is an increasing function. Hence \( y(t) \) is unbounded on \( [a, \infty) \).

**Theorem 3.3.** If \( p(t) \leq 0 \) and \( 2q(t) - p'(t) \geq d > 0 \), then there exists a nontrivial solution \( y(t) \) of \( (L) \) such that \( y(t) > 0, \ y'(t) < 0, \ y''(t) > 0 \), for all \( t \geq a \) and \( \lim_{t \to \infty} y(t) = \lim_{t \to \infty} y''(t) = 0 \). If \( w(t) \) is any bounded solution of \( (L) \), then for some \( K \), \( w(t) = Ky(t) \).

**Proof.** Let \( z_0(t), z_1(t) \) and \( z_2(t) \) be the three linearly independent solutions of \( (L) \). For each integer \( n > a \) there exist numbers \( c_{0n}, c_{1n} \) and \( c_{2n} \) such that

\[
\begin{align*}
c_{0n}z_0(n) + c_{1n}z_1(n) + c_{2n}z_2(n) &= 0, \\
c_{0n}z_0'(n) + c_{1n}z_1'(n) + c_{2n}z_2'(n) &= 0, \\
c_{0n}z_0''(n) + c_{1n}z_1''(n) + c_{2n}z_2''(n) &= 0 \\
\end{align*}
\]

(7)

\[
c_{0n}^2 + c_{1n}^2 + c_{2n}^2 = 1.
\]

Let \( y_n(t) \) be the solution of \( (L) \) defined by

\[
y_n(t) = c_{0n}z_0(t) + c_{1n}z_1(t) + c_{2n}z_2(t).
\]

By the independence of the solutions \( z_0, z_1 \) and \( z_2 \) and from (7), \( y_n(t) \) is a nontrivial solution of \( (L) \) for which

\[
y_n(n) = y_n'(n) = 0 \quad \text{and} \quad y_n''(n) > 0.
\]
Thus by Lemma 3.1 it follows that

\[(8) \quad y_n(t) > 0, \quad y'_n(t) < 0 \quad \text{and} \quad y''_n(t) > 0 \quad \text{for all} \quad t \in [a, b).
\]

By (7) there exists a sequence of integers \( \{n_i\} \) and numbers \( c_i, i = 0, 1, 2, \) such that \( \lim_{n_i \to \infty} c_{in_i} = c_i. \) Let \( y(t) \) be the solution of (L) defined by

\[(9) \quad y(t) = c_0z_0(t) + c_1z_1(t) + c_2z_2(t).
\]

From the independence of \( z_0, z_1 \) and \( z_2 \) and

\[(10) \quad c_0 + c_1 + c_2 = 1,
\]

it follows that \( y(t) \) is a nontrivial solution of (L). Since the sequences \( \{y_{n_0}(t)\}, \{y'_{n_1}(t)\} \) and \( \{y''_{n_2}(t)\} \) converge to the functions \( y(t), y'(t) \) and \( y''(t) \) respectively on any finite subinterval of \([a, \infty)\), it follows from (8) that

\[(11) \quad y(t) \geq 0, \quad y'(t) \leq 0 \quad \text{and} \quad y''(t) \geq 0 \quad \text{for} \quad t \in [a, \infty).
\]

If equality held at a point \( t \) in the first inequality of (11), then \( y(t) = 0 \) for \( t \in [t, \infty) \) which contradicts (9) and (10). Thus \( y(t) > 0, t \in [a, \infty) \).

Similarly, \( y'(t) < 0 \) and \( y''(t) > 0 \) for all \( t \in [a, \infty) \). Since \( y(t) \) is bounded by Lemma 3.2, \( F[y(t)] < 0 \) for \( t > a. \) Hence \( \int_a^t y(t)dt < \infty. \)

Since \( y'(t) \) is bounded, \( y(t) \to 0 \) as \( t \to \infty, \) and by Lemmas 1.1 and 1.3, \( y^{(k)}(t) \to 0 \) as \( t \to \infty, \) \( k = 0, 1, 2, 3. \)

To prove the last part of the theorem, let \( w(t) \) be a bounded solution of (L), and let \( K \) be a number such that \( w(b) - Ky(b) = 0. \) Consider the solution \( Z(t) = w(t) - Ky(t). \) If \( Z(t) \) were not identically equal to zero, then \( F[Z(b)] \geq 0, \) and it would follow from Lemma 3.2 that \( Z(t) \) could not be bounded, contradicting the boundedness of \( w(t) \) and \( y(t). \) This contradiction proves that \( Z(t) \) is the trivial solution of (L) and hence \( w(t) = Ky(t). \)

**References**


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