A NORM ON SEPARABLE NONCOMMUTATIVE
JORDAN ALGEBRAS OF DEGREE 2

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In [2] McCrimmon defined a norm on a nonassociative algebra $A$
defined on a vector space $X$ over a field $\Phi$ and with identity $e$ to be a
nondegenerate form $Q$ on $X$ for which there exist two rational mapp-
ings $E : x \rightarrow Ex, \ F : x \rightarrow Fx$ of $X$ into $\text{Hom}(X, X)$ and two rational
functions $e, \ f$ of $X$ into $\Phi$ satisfying

(a) $Ex = Fx = I$.

(b) $\partial_u E|_e = \alpha L_u, \ \partial_u F|_e = \beta R_u$ for some nonzero $\alpha, \ \beta \in \Phi$, where $\partial_u$ is
partial derivation in the direction $u \in X$ and where $L_u$ and $R_u$ are
respectively left and right multiplication in $A$ by $u$.

(c) $Q(Exy) = e(x)Q(y), \ Q(Fxy) = f(x)Q(y)$ whenever all mappings
involved are defined.

In that paper McCrimmon established the nice result that a
normed algebra is necessarily a separable noncommutative Jordan
algebra, and hence a direct sum of simple summands which are either
commutative Jordan algebras, quasi-associative algebras, or algebras
of degree 2. He pointed out that conversely any separable commuta-
tive Jordan algebra or quasi-associative algebra has a norm. How-
ever, it was not known at that time whether every separable non-
commutative Jordan algebra of degree 2 possessed a norm. The pur-
pose of this note is to exhibit a form $Q$ and associated mappings $E, F, \ e, \ f$ which make $Q$ a norm on any separable noncommutative Jordan
algebra of degree 2.

**Theorem.** Let $A$ be a separable noncommutative Jordan algebra of
degree 2 with identity $e$, and let $Q(x) = xx$ for all $x \in A$ where $x \rightarrow x$ is the
natural involution in $A$. Then

(1) $Q$ is a norm for $A$ using $Ex = L_x(L_x^{-1})^{-1}, \ Fx = R_x(R_x^{-1})^{-1}$, and
$e(x) = f(x) = Q^2(x)$.

(2) For each $x$ for which $Ex$ exists, the following are equivalent:

(i) $Fx$ exists,

(ii) $E^{x\rightarrow}$ exists,

(iii) $Ex$ belongs to the structure group $\Gamma(A)$ of $A$.

**Proof.** To establish part (1) we must show that properties (a)–(c)
hold. The first of these is obvious. For the second we need to define

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the mapping \( P_z = L_z(L_z + R_z) - L_z \), which satisfies \( L_z^{-1} = R_z P_z^{-1} \) for invertible \( z \) [1, p. 90, equation 1.2]. Thus \( (L_z^{-1})^{-1} R_z = P_z \) and \( E_z R_z = L_z(L_z^{-1})^{-1} R_z = L_z P_z \) whenever \( E_z \) is defined. Differentiating the last equation at \( c \) and in the direction \( u \) gives

\[
\partial_u (E_z R_z) |_c = \partial_u E_z |_c R_e + E_e \partial_u R_e |_c = \partial_u E_z |_c + R_u
\]

\[
= \partial_u (L_z P_z) |_c = \partial L_z |_c P_e + L_e \partial_u P_z |_c = L_u + L_u + R_u,
\]

using the relation \( \partial_u P_z |_c = L_u + R_u \) which is immediate from the definition of \( P_z \). Hence, \( \partial_u E_z |_c = 2L_u \). Since \( F_z \) is just obtained from \( E_z \) by interchanging left and right in \( A \), it is clear that \( \partial_u F_z |_c = 2R_u \).

Next we define an inner product on \( A \) by \( (y, z) = Q(y + z) - Q(y) - Q(z) = yz + zy \), and we claim that \( (y, xz) = (xy, z) \) holds identically in \( A \). This is equivalent to the relation \( y(xz) + (zx)y = (yx)z + z(xy) \), which is clearly true if any one of \( x, y, z \) is taken to be a multiple of \( c \). By linearity it is then sufficient to consider the case when all of \( x, y, z \) are skew, in which case the relation reduces to \( -y(xz) + (zx)y = -(yx)z + z(xy) \), which follows from the flexible law. Denoting the adjoint of any operator \( T \) with respect to this inner product by \( T' \), we have just shown that \( L_z' = L_z \).

We may now compute that

\[
Q(E_z y) = \frac{1}{2} (E_z y, E_z y) = \frac{1}{2} (L_z(L_z^{-1})^{-1} y, L_z(L_z^{-1})^{-1} y)
\]

\[
= \frac{1}{2} ((L_z'^{-1})^{-1} L_z L_z'(L_z'^{-1})^{-1} y, y).
\]

Thus in order to establish \( Q(E_z y) = Q^2(x) Q(y) \), it is sufficient to show that \( (L_z'^{-1})^{-1} L_z L_z'(L_z'^{-1})^{-1} = Q^2(x) I \). Since

\[
x^{-1} = \frac{1}{Q(x)} (\bar{x} x) x^{-1} = \frac{1}{Q(x)} \bar{x}
\]

and since \( L_z L_z' = L_z L_z \), this relation follows from

\[
(L_z'^{-1}) L_z L_z'(L_z'^{-1})^{-1} = \left( \frac{1}{Q(x)} L_z' \right)^{-1} L_z L_z \left( \frac{1}{Q(x)} L_z' \right)^{-1}
\]

\[
= Q^2(x) (L_z^{-1} L_z L_z'^{-1}) = Q^2(x) I.
\]

By left-right symmetry, it follows that \( Q(F_z y) = Q^2(x) Q(y) \) also holds.

To prove the second part of the theorem, we note first that the existence of \( E_z \) is equivalent to the conditions that \( x^{-1} \) exist and that \( L_x^{-1} \) be invertible. Suppose first that \( F_x \) exists and that \( E_x^{-1} \) does not exist. Then \( R_x^{-1} \) is invertible and there exists a nonzero \( w \in A \) such that \( 0 = xw = L_x w \). The flexible law gives \( x^{-1}(xw) + w(xx^{-1}) = (x^{-1}x)w + (wx)x^{-1} \), which reduces to \( 0 = (wx)x^{-1} \), or \( 0 = wx \) using

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the invertibility of $R_{x^{-1}}$. Hence, defining $\tau: A \to \Phi$ by $\tau(y)c = y + \hat{y}$, we have

$$0 = [wx]^{-1} = xw = [\tau(x)c - x][\tau(w)c - w] = \tau(x)\tau(y)c - \tau(w)x - \tau(x)w,$$

showing that $\tau(x)w$ is in the subalgebra generated by $x$. But there are no zero divisors in this subalgebra since $x$ is invertible, so $\tau(x)w = 0$. Therefore, since $L_{x^{-1}}$ is invertible, $0 \neq x^{-1}w = Q(x)xw = Q(x)[\tau(x)1 - x]$ \(w = 0\). This contradiction shows that (i) implies (ii).

Suppose next that $E_{x^{-1}}$ exists and $F_x$ does not exist. Then $L_x$ is invertible and there exists a nonzero $w \in A$ such that $wx^{-1} = 0$. The flexible law gives $x(x^{-1}w) + w(x^{-1}x) = (xx^{-1})w + (wx^{-1})x$, which reduces this time to $x(x^{-1}w) = 0$. But since $L_x$ and $L_{x^{-1}}$ are invertible, $w = 0$ contrary to assumption. Thus (ii) implies (i).

Assume now that (i) and (ii) hold. Then $E_{x^{-1}}$ is clearly the inverse of $E_x$, so $E_x$ is invertible, and by symmetry, $F_x$ is also invertible. Next we recall that a linear transformation $W$ is in the structure group $\Gamma(A)$ if and only if it is invertible and if there exists an invertible linear transformation $V$ so that $(Wy)^{-1} = F^{-1}V^{-1}$ for all invertible $y \in A$. In our case, setting $z = (L_{x^{-1}})^{-1}y$ or $y = x^{-1}z$, we have

$$Q(E_{x}y)(E_{x}y)^{-1} = [E_{x}y]^{-1} = [L_{x}(L_{x^{-1}})^{-1}]^{-1} = \hat{z}z = R_{z}(R_{z^{-1}})^{-1} = (Q(y)y^{-1})^{-1} = Q^{2}(x)Q(y)R_{z^{-1}}(R_{z})^{-1}y^{-1}.$$  

Since $x$ and $y$ are invertible, $Q(x)$ and $Q(y)$ are nonzero and so this reduces to $(E_{x}y)^{-1} = (F_{x})^{-1}y^{-1}$, establishing (iii).

And finally, if (iii) holds then $E_{x}$ is necessarily invertible. Hence $(E_{x})^{-1} = E_{x^{-1}}$ exists and (ii) holds.

**References**


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