A NORM ON SEPARABLE NONCOMMUTATIVE JORDAN ALGEBRAS OF DEGREE 2

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In [2] McCrimmon defined a norm on a nonassociative algebra A defined on a vector space X over a field Φ and with identity c to be a nondegenerate form Q on X for which there exist two rational mappings $E: x \to E_x$, $F: x \to F_x$ of X into $\mathrm{Hom}(X, X)$ and two rational functions e, g of X into Φ satisfying

(a) $E_c = F_c = I$.

(b) $\partial_u E|_c = \alpha L_u$, $\partial_u F|_c = \beta R_u$ for some nonzero $\alpha, \beta \in \Phi$, where ∂_u is partial derivation in the direction $u \in X$ and where L_u and R_u are respectively left and right multiplication in A by u.

(c) $Q(E_x y) = e(x)Q(y)$, $Q(F_x y) = f(x)Q(y)$ whenever all mappings involved are defined.

In that paper McCrimmon established the nice result that a normed algebra is necessarily a separable noncommutative Jordan algebra, and hence a direct sum of simple summands which are either commutative Jordan algebras, quasi-associative algebras, or algebras of degree 2. He pointed out that conversely any separable commutative Jordan algebra or quasi-associative algebra has a norm. However, it was not known at that time whether every separable noncommutative Jordan algebra of degree 2 possessed a norm. The purpose of this note is to exhibit a form Q and associated mappings E, F, e, f which make Q a norm on any separable noncommutative Jordan algebra of degree 2.

THEOREM. *Let A be a separable noncommutative Jordan algebra of degree 2 with identity c, and let $Q(x) = \bar{x}x$ for all $x \in A$ where $x \to \bar{x}$ is the natural involution in A. Then*

(1) *Q is a norm for A using $E_x = L_x(L_{x^{-1}})^{-1}$, $F_x = R_x(R_{x^{-1}})^{-1}$, and $e(x) = f(x) = Q^2(x)$.*

(2) *For each x for which E_x exists, the following are equivalent:*

(i) *F_x exists,*

(ii) *$E_{x^{-1}}$ exists,*

(iii) *E_x belongs to the structure group $\Gamma(A)$ of A.*

PROOF. To establish part (1) we must show that properties (a)–(c) hold. The first of these is obvious. For the second we need to define

Received by the editors October 26, 1967.

[1] This research was supported by NSF Grant GP-3993.

the mapping $P_x = L_x(L_x+R_x) - L_{x^2}$, which satisfies $L_{x^{-1}} = R_x P_x^{-1}$ for invertible x [1, p. 90, equation 1.2]. Thus $(L_{x^{-1}})^{-1}R_x = P_x$ and $E_x R_x = L_x(L_{x^{-1}})^{-1}R_x = L_x P_x$ whenever E_x is defined. Differentiating the last equation at c and in the direction u gives

$$\begin{aligned}\partial_u(E_xR_x)\big|_c &= \partial_u E_x\big|_c R_c + E_c\partial_u R_x\big|_c = \partial_u E_x\big|_c + R_u \\ &= \partial_u(L_xP_x)\big|_c = \partial L_x\big|_c P_c + L_c\partial_u P_x\big|_c = L_u + L_u + R_u,\end{aligned}$$

using the relation $\partial_u P_x\big|_c = L_u + R_u$ which is immediate from the definition of P_x. Hence, $\partial_u E_x\big|_c = 2L_u$. Since F_x is just obtained from E_x by interchanging left and right in A, it is clear that $\partial_u F_x\big|_c = 2R_u$.

Next we define an inner product on A by $(y,z) = Q(y+z) - Q(y) - Q(z) = \bar{y}z + \bar{z}y$, and we claim that $(y, xz) = (\bar{x}y, z)$ holds identically in A. This is equivalent to the relation $\bar{y}(xz) + (\bar{z}\bar{x})y = (\bar{y}x)z + \bar{z}(\bar{x}y)$, which is clearly true if any one of x, y, z is taken to be a multiple of c. By linearity it is then sufficient to consider the case when all of x, y, z are skew, in which case the relation reduces to $-y(xz) + (zx)y = -(yx)z + z(xy)$, which follows from the flexible law. Denoting the adjoint of any operator T with respect to this inner product by T', we have just shown that $L_x' = L_{\bar{x}}$.

We may now compute that

$$\begin{aligned}Q(E_xy) &= \tfrac{1}{2}(E_xy, E_xy) = \tfrac{1}{2}(L_x(L_{x^{-1}})^{-1}y, L_x(L_{x^{-1}})^{-1}y) \\ &= \tfrac{1}{2}((L'_{x^{-1}})^{-1}L_{\bar{x}}L_x(L_{x^{-1}})^{-1}y, y).\end{aligned}$$

Thus in order to establish $Q(E_xy) = Q^2(x)Q(y)$, it is sufficient to show that $(L'_{x^{-1}})^{-1}L_{\bar{x}}L_x(L_{x^{-1}})^{-1} = Q^2(x)I$. Since

$$x^{-1} = \frac{1}{Q(x)}(\bar{x}x)x^{-1} = \frac{1}{Q(x)}\bar{x}$$

and since $L_{\bar{x}}L_x = L_xL_{\bar{x}}$, this relation follows from

$$\begin{aligned}(L'_{x^{-1}})L_{\bar{x}}L_x(L_{x^{-1}})^{-1} &= \left(\frac{1}{Q(x)}L'_{\bar{x}}\right)^{-1}L_xL_{\bar{x}}\left(\frac{1}{Q(x)}L_{\bar{x}}\right)^{-1} \\ &= Q^2(x)(L_x)^{-1}L_xL_{\bar{x}}(L_{\bar{x}})^{-1} = Q^2(x)I.\end{aligned}$$

By left-right symmetry, it follows that $Q(F_xy) = Q^2(x)Q(y)$ also holds.

To prove the second part of the theorem, we note first that the existence of E_x is equivalent to the conditions that x^{-1} exist and that $L_{x^{-1}}$ be invertible. Suppose first that F_x exists and that $E_{x^{-1}}$ does not exist. Then $R_{x^{-1}}$ is invertible and there exists a nonzero $w \in A$ such that $0 = xw = L_xw$. The flexible law gives $x^{-1}(xw) + w(xx^{-1}) = (x^{-1}x)w + (wx)x^{-1}$, which reduces to $0 = (wx)x^{-1}$, or $0 = wx$ using the invertibility of $R_{x^{-1}}$. Hence, defining $\tau: A \to \Phi$ by $\tau(y)c = y + \bar{y}$, we have

$$0 = [wx]^- = \bar{x}\bar{w} = [\tau(x)c - x][\tau(w)c - w] = \tau(x)\tau(y)c - \tau(w)x - \tau(x)w,$$

showing that $\tau(x)w$ is in the subalgebra generated by x. But there are no zero divisors in this subalgebra since x is invertible, so $\tau(x)w = 0$. Therefore, since $L_{x^{-1}}$ is invertible, $0 \neq x^{-1}w = Q(x)\bar{x}w = Q(x)[\tau(x)1 - x]w = 0$. This contradiction shows that (i) implies (ii).

Suppose next that $E_{x^{-1}}$ exists and F_x does not exist. Then L_x is invertible and there exists a nonzero $w \in A$ such that $wx^{-1} = 0$. The flexible law gives $x(x^{-1}w) + w(x^{-1}x) = (xx^{-1})w + (wx^{-1})x$, which reduces this time to $x(x^{-1}w) = 0$. But since L_x and $L_{x^{-1}}$ are invertible, $w = 0$ contrary to assumption. Thus (ii) implies (i).

Assume now that (i) and (ii) hold. Then $E_{x^{-1}}$ is clearly the inverse of E_x, so E_x is invertible, and by symmetry, F_x is also invertible. Next we recall that a linear transformation W is in the structure group $\Gamma(A)$ if and only if it is invertible and if there exists an invertible linear transformation V so that $(Wy)^{-1} = V^{-1}y^{-1}$ for all invertible $y \in A$. In our case, setting $z = (L_{x^{-1}})^{-1}y$ or $y = x^{-1}z$, we have

$$\begin{aligned} Q(E_xy)(E_xy)^{-1} &= [E_xy]^- \\ &= [L_x(L_{x^{-1}})^{-1}y]^- = [xz]^- = \bar{z}\bar{x} \\ &= R_{\bar{x}}(R_{\bar{x}^{-1}})^{-1}(\bar{z}\bar{x}^{-1}) = R_{\bar{x}}(R_{\bar{x}^{-1}})^{-1}\bar{y} \\ &= R_{Q(x)x^{-1}}(R_{[Q(x)x^{-1}]^{-1}})^{-1}(Q(y)y^{-1}) = Q^2(x)Q(y)R_{x^{-1}}(R_x)^{-1}y^{-1}. \end{aligned}$$

Since x and y are invertible, $Q(x)$ and $Q(y)$ are nonzero and so this reduces to $(E_xy)^{-1} = (F_x)^{-1}y^{-1}$, establishing (iii).

And finally, if (iii) holds then E_x is necessarily invertible. Hence $(E_x)^{-1} = E_{x^{-1}}$ exists and (ii) holds.

References

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