A CLASS OF RELATED DIRICHLET AND INITIAL VALUE PROBLEMS

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1. **Introduction.** In some recent papers ([1], [2], [3]), the authors have exhibited a number of relationships existing among solution pairs of related initial-boundary value problems. In particular, it was shown, under suitable growth restrictions, that the solution of an initial value problem for a wave type equation is obtainable from the solution of an associated heat type equation by means of the inverse Laplace transform. The main interest in this paper is to obtain analogous results connecting solutions of Dirichlet type problems for half spaces to solutions of initial value heat type problems. For precision, let $x=(x_1, x_2, \cdots, x_n)$, let $D=(D_1, D_2, \cdots, D_n)$ with $D_i=\partial/\partial x_i$, and let $P(x, D)$ be a finite order linear partial differential operator (usually elliptic type). Then consider the following pair of problems:

$$(\mathrm{P}_1) \qquad \begin{aligned} &(\mathrm{a}) \qquad u_t(x, t) = P(x, D)u(x, t), \qquad t > 0, \\ &(\mathrm{b}) \qquad u(x, 0) = \phi(x) \end{aligned}$$

and

$$(\mathrm{P}_2) \qquad \begin{aligned} &(\mathrm{a}) \qquad v_{yy}(x, y) + P(x, D)v(x, y) = 0, \qquad y > 0, \\ &(\mathrm{b}) \qquad v(x, 0) = \phi(x). \end{aligned}$$

In §2, we shall show under suitable restrictions on $\phi(x)$ and $u(x, t)$, a solution of (P_1), that

$$v(x, y) = \frac{y}{\sqrt{\pi}} \int_0^\infty \exp[-y^2\xi]\xi^{-1/2}u(x, 1/4\xi)d\xi \tag{1.1}$$

is a solution of (P_2). Furthermore, we shall show in §3 that under appropriate conditions relation (1.1) can be inverted to give

$$u(x, t) = \frac{\sqrt{\pi}}{2} t^{-1/2}\mathfrak{L}_s^{-1}\{s^{-1/2}v(x, s^{1/2})\}_{s\to 1/4t}, \tag{1.2}$$

a solution of (P_1) in terms of $v(x, y)$, a solution of (P_2). §4 will deal with these results in the case where $(\mathrm{P}_1\mathrm{a})$ is the standard heat equation with $n=1$ and (P_2) is the half-plane Dirichlet problem. In §5 we

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conclude the paper with some general remarks about fundamental solutions and harmonic functions.

2. **Solutions of** (P_2) **from solutions of** (P_1). In this section we prove two theorems showing how to obtain solutions of (P_2) from solutions of (P_1).

THEOREM 2.1. *Let* $u(x, t)$ *denote a solution of* (P_1) *satisfying condition* C^α: $|u(x, t)-\phi(x)| \leqq Mt^\alpha$ *almost everywhere, where* M *is a positive constant,* $0<\alpha<1/2$, *and* $\phi(x)$ *is continuous. Then a solution of* (P_2) *is given by* (1.1) *and*

$$|v(x, y) - \phi(x)| \leqq \frac{M\Gamma(1/2 - \alpha)}{4^\alpha\sqrt{\pi}} y^{2\alpha}. \tag{2.1}$$

PROOF. Let R be an arbitrary positive number and restrict x to the range $|x| \leqq R$. Under a change of variables, (1.1) can be expressed in the form

$$v(x, y) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\xi}\xi^{-1/2}u(x, y^2/4\xi)d\xi. \tag{2.2}$$

We first show that (P_2b) and (2.1) hold. From the condition C^α and the fact that $\phi(x) = \pi^{-1/2}\int_0^\infty e^{-\xi}\xi^{-1/2}\phi(x)d\xi$, we obtain

$$\begin{aligned} |v(x, y) - \phi(x)| &\leqq \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\xi}\xi^{-1/2}|u(x, y^2/4\xi) - \phi(x)|d\xi \\ &\leqq \frac{My^{2\alpha}}{4^\alpha\sqrt{\pi}}\int_0^\infty e^{-\xi}\xi^{-(1/2+\alpha)}d\xi = \frac{M\Gamma(1/2 - \alpha)}{4^\alpha\sqrt{\pi}} y^{2\alpha}. \end{aligned}$$

Then since $\alpha > 0$, $\lim_{y\to 0} v(x, y) = \phi(x)$.

We next show that (P_2a) is satisfied by $v(x, y)$ for $y > 0$. From (2.2) and the fact that $u_y(x, y^2/4\xi) = -(2\xi/y)u_\xi(x, y^2/4\xi)$, we obtain formally

$$v_y(x, y) = \frac{-2}{y\sqrt{\pi}}\int_0^\infty \xi^{1/2}e^{-\xi}u_\xi(x, y^2/4\xi)d\xi. \tag{2.3}$$

An integration by parts and an application of condition C^α permits us to obtain

$$v_y(x, y) = \frac{2}{y\sqrt{\pi}}\int_0^\infty e^{-\xi}(-\xi^{1/2} + 1/2\xi^{-1/2})u(x, y^2/4\xi)d\xi \tag{2.4}$$

and

$$|v_y(x, y)| \leqq \frac{2}{y\sqrt{\pi}} \int_0^\infty e^{-\xi}(\xi^{1/2} + 1/2\xi^{-1/2}) \left(|\phi(x)| + \frac{My^{2\alpha}}{4^\alpha} \xi^{-\alpha} \right) d\xi.$$

With our restrictions on x, ($|\phi(x)|$ is bounded) and α, the integral in the right member of this inequality converges uniformly for $y \geqq \delta > 0$. This proves that $v_y(x, y)$ exists for $y > 0$. The relation (2.4) can be rewritten in the form

$$v_y(x, y) = \frac{v}{y} - \frac{2}{y\sqrt{\pi}} \int_0^\infty \xi^{1/2} e^{-\xi} u(x, y^2/4\xi) d\xi. \tag{2.5}$$

A further differentiation of this with respect to y along with a reduction using (2.5) yields

$$v_{yy}(x, y) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \xi^{-1/2} e^{-\xi} u_2(x, y^2/4\xi) d\xi \tag{2.6}$$

where the subscript 2 denotes differentiation with respect to $y^2/4\xi$. From the fact that $u_2(x, y^2/4\xi) = -(4\xi^2/y^2) u_\xi(x, y^2/4\xi)$ and our previous argument, $v_{yy}(x, y)$ exists for $y > 0$. But $u_2(x, y^2/4\xi) = P(x, D) \cdot u(x, y^2/4\xi)$ from (P_1a), and (2.6) reduces to $v_{yy}(x, y) = -P(x, D) v(x, y)$ for $y > 0$. Since R was arbitrary, this completes the proof.

THEOREM 2.2. *Let $u(x, t)$ denote a bounded continuous solution of* (P_1) *corresponding to bounded continuous $\phi(x)$. If $\lim_{t\to 0^+} u(x, t) = \phi(x)$ pointwise, a solution of* (P_2) *is given by* (1.1) *with $v(x, y)$ bounded.*

PROOF. Our restriction on x in the proof of Theorem 2.1 is no longer necessary. From (2.2) and (2.4) and the fact that $u(x, t)$ is bounded, it is clear that $v(x, y)$ is bounded and that $v_y(x, y)$ exists for $y > 0$. Similarly $v_{yy}(x, y)$ exists and is given by (2.6). Hence (P_2a) is satisfied by $v(x, y)$. It remains to be shown that $v(x, y)$ satisfies (P_2b) at arbitrary x_0. Let $N = \max |u(x, t) - \phi(x)|$ (if $N = 0$, the proof is trivial) and let $\epsilon > 0$ be arbitrary. Select $K = \pi\epsilon^2/4N^2$ and choose $\delta > 0$ such that $|u(x_0, y^2/K) - \phi(x_0)| < \epsilon/2$ for $0 < y < \delta$. We have

$$\begin{aligned} &|v(x_0, y) - \phi(x_0)| \\ &\leqq \frac{1}{\sqrt{\pi}} \left\{ \int_0^K + \int_K^\infty e^{-\xi} \xi^{-1/2} |u(x_0, y^2/4\xi) - \phi(x_0)| d\xi \right\}. \end{aligned} \tag{2.7}$$

Our restriction on y shows the second integral in (2.7) is bounded by $\epsilon/2$ while the first integral is bounded by $\epsilon/2$ (by replacing the factor $e^{-\xi}$ by 1 before integrating). This shows that $|v(x_0, y) - \phi(x_0)| \leqq \epsilon$ and completes the proof.

3. **Solutions of** (P_1) **from solutions of** (P_2). The conditions required for applying the standard Laplace inversion theorems are too restrictive to prove the invertability of (1.1) in every case. The boundedness assumption on $|v(x, y)|$ at most permits one to assert that the corresponding solution $u(x, t)$ of (P_1) exists in the sense of distributions [**4**, p. 236]. There are many situations in which the function $s^{-1/2}v(x, s^{1/2})$ can either be inverted directly or at least exhibits suitable properties for its invertability in the classical sense. For these cases, we prove the following result:

THEOREM 3.1. *Let $v(x, y)$ be the solution of a well-posed problem* (P_2) *for continuous data $\phi(x)$. If the inverse Laplace transforms of $s^{-1/2}v(x, s^{1/2})$ and $s^{-1/2}v_s(x, s^{1/2})$ exist in the classical sense and the function $u(x, t)$ defined by* (1.2) *is a bounded differentiable function, then $u(x, t)$ is a solution of* (P_1).

PROOF. Theorem 2.2 shows that $u(x, t)$ satisfies (P_1b), for if $u(x, 0+) \neq \phi(x)$, we could apply the theorem to construct a solution of (P_2) distinct from the given $v(x, y)$ contrary to well-posedness. To show that $u(x, t)$ satisfies (P_1a), introduce the change of variables $\tau = 1/4t$ and let $W(x, \tau) = (1/\sqrt{\pi})u(x, 1/4\tau)$. The following standard results for Laplace transforms will be needed:

$$\text{(3.1)} \qquad \begin{aligned} &\text{(a)} \qquad \tau \mathcal{L}_s^{-1}\{f(s)\}_{s\to\tau} = \mathcal{L}_s^{-1}\{-\partial/\partial s f(s)\}_{s\to\tau}, \\ &\text{(b)} \qquad \mathcal{L}_s^{-1}\{s^{-1}f(s)\}_{s\to\tau} = \int_0^\tau F(\xi)d\xi, \end{aligned}$$

where $F(\tau) = \mathcal{L}_s^{-1}\{f(s)\}_{s\to\tau}$. The notation $s \to \tau$ will be omitted in the inverse transforms below.

From (1.3) we have

$$\text{(3.2)} \qquad \begin{aligned} W(x, \tau) &= \tau^{-1/2}[\tau \mathcal{L}_s^{-1}\{s^{-1/2}v(x, s^{1/2})\}] \\ &= \tau^{-1/2}\mathcal{L}_s^{-1}\left\{\frac{-\partial}{\partial s}(s^{-1/2}v)\right\} \\ &= \tau^{-1/2}[\mathcal{L}_s^{-1}\{\tfrac{1}{2}s^{-3/2}v - s^{-1/2}v_s\}] \\ &= \tfrac{1}{2}\tau^{-1/2}\mathcal{L}_s^{-1}\{s^{-1}[s^{-1/2}v]\} - \tau^{-1/2}\mathcal{L}_s^{-1}\{s^{-1/2}v_s\}. \end{aligned}$$

But the definition of $W(x, \tau)$ and (3.1b) give

$$\text{(3.3)} \qquad \mathcal{L}_s^{-1}\{s^{-1}[s^{-1/2}v]\} = \int_0^\tau \xi^{-1/2}W(x, \xi)d\xi.$$

It follows from this and (3.2) that

$$W(x, \tau) - \tfrac{1}{2}\tau^{-1/2}\int_0^\tau \xi^{-1/2}W(x, \xi)d\xi = -\tau^{-1/2}\mathcal{L}_s^{-1}\{s^{-1/2}v_s\}. \tag{3.4}$$

Upon multiplying both sides of this by $\tau^{3/2}$, reapplying (3.1a) to the right member, and using the fact that $v_{ss}(x, s^{1/2}) = v_{22}/4s - v_s/2s$, we get (after using (3.4) to replace $\mathcal{L}_s^{-1}\{s^{-3/2}v_s\}$)

$$\begin{aligned} \tau^{3/2}W(x, \tau) - \frac{1}{2}\tau\int_0^\tau \xi^{-1/2}W(x, \xi)d\xi \\ = -\frac{1}{2}\int_0^\tau\int_0^\xi \eta^{-1/2}W(x, \eta)d\eta d\xi \\ + \int_0^\tau \xi^{1/2}W(x, \xi)d\xi + \tfrac{1}{4}\mathcal{L}_s^{-1}\{s^{-3/2}v_{22}(x, s^{1/2})\}. \end{aligned} \tag{3.5}$$

The last term in this is just

$$-\tfrac{1}{4}\mathcal{L}_s^{-1}\{s^{-3/2}P(x, D)v(x, s^{1/2})\} = -\tfrac{1}{4}P(x, D)\int_0^\tau \xi^{-1/2}W(x, \xi)d\xi$$

by (P_2a) and (3.3). Differentiation of (3.5) with respect to τ produces the relation

$$4\tau^2 W_\tau(x, \tau) = -P(x, D)W(x, \tau)$$

and this is just (P_1a) after a return to the original variables t and u. This completes the proof.

4. The half-plane Dirichlet problem. The condition C^α was used throughout the proof of Theorem 2.1 to guarantee the existence of certain differentiated integrals. If $n=1$ and $P(x, D) = D^2$, then problem (P_1) is the Cauchy problem for the standard heat equation. The solution is given by

$$u(x, t) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t}\phi(\xi)d\xi. \tag{4.1}$$

A Hölder condition of order γ, $0<\gamma<1$, on the data function $\phi(x)$ suffices to show that $u(x, t)$ satisfies condition C^α. Since $r^\gamma e^{-\beta r^2} \leqq (\gamma/2\beta)^{\gamma/2}e^{-\gamma/2}$, it follows that if $|\phi(x) - \phi(\xi)| \leqq K|x-\xi|^\gamma$, then

$$|\phi(x) - \phi(\xi)|\, e^{-|x-\xi|^2/8t} \leqq K|x-\xi|^\gamma e^{-|x-\xi|^2/8t} \leqq K(4\gamma/e)^{\gamma/2}t^{\gamma/2}.$$

Hence,

$$\begin{aligned}|u(x,t)-\phi(x)| &\leqq \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-|x-\xi|^2/8t}\,|\phi(x)-\phi(\xi)|\,e^{-|x-\xi|^2/8t}d\xi \\ &\leqq \sqrt{2}K(4\gamma/e)^{\gamma/2}t^{\gamma/2}.\end{aligned}$$

In this case, the problem (P_2) is the half-plane Dirichlet problem. Within the class of solutions satisfying (2.1) the solution is unique. This follows from a consideration of $w(x, y)$, the difference of two possible solutions, which is harmonic in the upper half-plane and vanishes on the real axis. By the Schwarz reflection principle $w(x, y)$ can be continued to the lower half-plane as the imaginary part of an entire function $f(z)$ which is real on the real axis. But $|w(x, y)| \leqq A|y|^{2\alpha} \leqq Ar^{2\alpha}$ and by a well-known result [**5**, p. 87] $f(z)$ is constant since $2\alpha<1$. Hence, $w(x, y)\equiv 0$, which establishes the uniqueness.

Our results (1.1) and (1.2), therefore, relate heat functions to harmonic functions and vice versa. This permits the use of the results of complex variables, such as conformal mappings, for treating heat problems (and, consequently, wave problems [**1**], [**2**], [**3**]). In this connection, it follows from the Cauchy-Riemann conditions that for $y>0$, the harmonic conjugate $v^*(x, y)$ of $v(x, y)$ is given by

$$v^*(x, y) = \text{const} - \frac{y}{2\sqrt{\pi}}\int_0^{\infty} \xi^{-3/2}e^{-\xi}u_x(x, y^2/4\xi)d\xi \tag{4.2}$$

provided that this integral exists.

5. **Further remarks.** For the heat equation, the condition $|\phi(x)| \leqq M$ implies $|u(x, t)| \leqq M$. From Theorem 2.2 we get $|v(x, y)| \leqq M$ and this is a maximum principle for the related Dirichlet problem. Consequently, if (P_1) satisfies such a condition, then Theorem 2.2 implies a maximum principle for the problem (P_2) with this type of data.

It is readily checked that the standard heat kernel in n space variables transforms into

$$v(x-\xi, y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\, y\left[\sum_{i=1}^{n}(x_i-\xi_i)^2+y^2\right]^{-(n+1)/2},$$

the Dirichlet kernel for the half-space $y>0$. If the operator $P(x, D)$ is elliptic and of the second order, the bounds developed for the kernel of the corresponding parabolic problem are available for discussing the kernel of the related problem (P_2) (see [**6**, Chapter 1]). In this situation, $\phi(x)$ is a point distribution and the condition C^α would fail to hold at the point of support of this distribution.

For treating problems of the type (P_1) and (P_2) with somewhat different boundaries, the reader is referred to the papers mentioned earlier.

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