SYMMETRIC OPERATORS WITH SINGULAR SPECTRAL FUNCTIONS

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In a previous article [3] it was shown that if \( A \) is a closed symmetric operator with deficiency indices \((1, 1)\) in the Hilbert space \( H \) and if \( \mu_0 \) is a point of regular type for \( A \), then there is a neighborhood of \( \mu_0 \) in which every minimal selfadjoint dilation \( A^+ \) of \( A \) has spectral multiplicity not more than 1. This shows that in Theorem 14 of Coddington [2] and Theorem 5.2 of McKelvey [6] it is the hypothesis that every point is of regular type which determines the spectral multiplicity of the dilation; the hypothesis that the contraction \( F(\lambda) \) is continuous down to the real axis and has norm less than 1 there serves to make the spectral function absolutely continuous. (See Remark 2.)

The procedure used in [3] depended on the fact that if \( \mu_0 \) is a point of regular type for \( A \), then there exists a selfadjoint extension \( A_0 \) of \( A \) for which \( \mu_0 \) is in the resolvent set. This means that for different points \( \mu_0 \) one might have to choose different operators \( A_0 \). In the present note this necessity is eliminated. It is shown that if \( A \) has a selfadjoint extension \( A_0 \) with a pure point spectrum with no finite limit points, then for each minimal selfadjoint extension or dilation \( A^+ \) of \( A \) there is defined on the real axis a nondecreasing function \( \rho(\sigma) \) (which depends solely on \( A^+ \) and on the single selfadjoint extension \( A_0 \)) such that \( A^+ \) is unitarily equivalent to the multiplication operator in \( L^2_\sigma \). Krein [5] has shown that if \( A \) is simple, then there exists a selfadjoint extension \( A_0 \) of the type described above if and only if every point is of regular type for \( A \). (See Remark 1 also.)

In the case that \( A \) is a singular Sturm-Liouville operator, the results of the paper are extended to the case that there exists a selfadjoint extension \( A_0 \) with a singular spectral function.

For terminology used we refer the reader to [3] and to Achieser and Glasmann [1].

We note first of all the following lemma due to Štraus [8].

**Lemma 1.** Suppose \( \Phi(\lambda) \) is analytic in the upper halfplane with nonnegative imaginary part. Suppose \( \Psi(\lambda) \) is analytic in some open set containing the real interval \([\alpha, \beta]\). Then,

Presented to the Society, March 4, 1968; received by the editors February 29, 1968.

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\[
(2\pi i)^{-1} \lim_{r \to +0} \int_{-\beta}^{\beta} \Phi(\sigma + ir)\Psi(\sigma + ir) - \Phi(\sigma + ir)\Psi(\sigma - ir) \, d\sigma
\]

\[
= \int_{-\beta}^{\beta} \Psi(\sigma)d\rho(\sigma), \quad \text{where} \quad \rho(\sigma) = (1/\pi) \lim_{r \to +0} \int_{0}^{\pi} I\Phi(\sigma + ir) \, d\sigma.
\]

Here \( I \) stands for the imaginary part and [ ] denotes complex conjugate.

We note further the following lemma due to Kreïn. (See Achieser and Glasmann [1, Appendix I, §4].)

**Lemma 2.** Suppose that \( A \) is a closed symmetric operator with deficiency indices \((1, 1)\) in the Hilbert space \( H \). Let \( A_0 \) be any self-adjoint extension of \( A \). Let \( g(\lambda) = g_0 + (\lambda - \lambda_0)R_0(\lambda)g_0 \), where \( \lambda_0 \) is a complex number with positive imaginary part; \( g_0 \) is an element of norm 1 in the deficiency subspace of \( A \) corresponding to \( \lambda_0 \); \( R_0(\lambda) \) is the resolvent of \( A_0 \). Let \( Q(\lambda) = i\lambda_0 + (\lambda - \lambda_0)(g_0, g(\lambda)) \). Then for \( \Im \lambda \neq 0 \) the generalized resolvent \( R(\lambda) \) of \( A \) corresponding to a self-adjoint extension or dilation \( A^+ \) of \( A \) has the form

\[
(1) \quad R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q_1(\lambda)]^{-1}(\cdot, g(\lambda))g(\lambda),
\]

where \( \theta(\lambda) \) is an analytic function for \( \Im \lambda \neq 0 \) which has nonnegative imaginary part in the upper half-plane. \( R(\lambda) \) corresponds to a self-adjoint extension in the space \( H \) if and only if \( \theta(\lambda) \) is identically equal to a constant \( \theta \), where \( -\infty < \theta < +\infty \). The case \( \theta = +\infty \) gives the resolvent \( R_0(\lambda) \) corresponding to \( A_0 \).

**Theorem.** Let \( A \) be a simple closed symmetric operator with deficiency indices \((1, 1)\) in the Hilbert space \( H \). Let \( A_0 \) be a self-adjoint extension of \( A \) with a pure point spectrum having no finite limit points. Let \( A^+ \) be a minimal self-adjoint extension of \( A \) with corresponding generalized resolvent of \( A \) represented as in Lemma 2. Then, \( A^+ \) is unitarily equivalent to the multiplication operator in \( L^2_p \), where

\[
(2) \quad \rho(\sigma) = (1/\pi) \lim_{r \to +0} \int_{0}^{\pi} I\Phi(\mu + ir) \, d\mu,
\]

\[
(3) \quad \Phi(\lambda) = [\theta(\lambda)Q_1(\lambda) - 1][\theta(\lambda) + Q_1(\lambda)]^{-1}.
\]

If \( E^+(\lambda) \) is the spectral function of \( A^+ \), then for all elements \( f, h \in H \),

\[
(4) \quad ([E^+(\beta) - E^+(\alpha)]f, h) = \int_{E(\alpha)}^{E(\beta)} (f, g(\sigma))(g(\sigma), h)[Q_1(\sigma) + 1]^{-1}d\rho(\sigma),
\]

where the integral is a Lebesgue-Stieltjes integral.
Proof. For any $f, g$ in $H$, Kren's formula (1) for the generalized resolvent $R(\lambda)$ corresponding to $A^+$ can be written in the form

$$ R(\lambda)f, h = F_1(\lambda) + F_2(\lambda)[\theta(\lambda)Q_1(\lambda) - 1][\theta(\lambda) + Q_1(\lambda)]^{-1}, $$

where

$$ F_1(\lambda) = \{(R_0(\lambda)f, h)[Q^*_1(\lambda) + 1] - Q_1(\lambda)(f, g(\lambda))(g(\lambda), h)\}[Q^*_1(\lambda) + 1]^{-1} $$

and

$$ F_2(\lambda) = (f, g(\lambda))(g(\lambda), h)[Q_1(\lambda) + 1]^{-1}. $$

We claim that $F_1(\lambda)$ and $F_2(\lambda)$ are entire functions of $\lambda$.

In order to establish this claim, we observe that the following equations are true:

1. $$(R_0(\lambda)f, h) = \int_0^\infty (\mu - \lambda)^{-1}d(E_0(\mu)f, h);$$
2. $$(f, g(\lambda)) = (f, g_0) + (\lambda - \lambda_0)\int_0^\infty (\mu - \lambda)^{-1}d(E_0(\mu)f, g_0);$$
3. $$(g(\lambda), h) = (g_0, h) + (\lambda - \lambda_0)\int_0^\infty (\mu - \lambda)^{-1}d(E_0(\mu)g_0, h);$$
4. $$Q_1(\lambda) = iD_{\lambda_0} + (\lambda - \lambda_0)$$
5. $$+ (\lambda - \lambda_0)(\lambda - \lambda_0)\int_0^\infty (\mu - \lambda)^{-1}d(E_0(\mu)g_0, g_0),$$

where $E_0(\mu)$ is the spectral function of $A_0$.

If $\mu_0$ is a point in the resolvent set of $A_0$, then each of the functions $(R_0(\lambda)f, h), (f, g(\lambda)), (g(\lambda), h), Q_1(\lambda)$ is an analytic function of $\lambda$ in a neighborhood of $\mu_0$. Further, $Q_1(\lambda)$ is real for $\lambda$ real, so that $Q^*_1(\lambda) + 1$ is not zero in this neighborhood for $\lambda$ sufficiently close to the real axis. It follows that $F_1(\lambda)$ and $F_2(\lambda)$ are analytic in a neighborhood of $\mu_0$.

Suppose $\mu_0$ is an eigenvalue of $A_0$. Since $A$ is simple, $g_0$ is a generating element for $A_0$. (See Achiesser and Glasmann [1, §81, Theorem 2].) Hence we may write equations (8), (9), (10), (11) in the form

$$ (R_0(\lambda)f, h) = G_1(\lambda) $$

$$ + (\mu_0 - \lambda)^{-1}(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)(E_0[\mu_0]g_0, g_0)^{-1}, $$

$$ (f, g(\lambda)) = G_2(\lambda) + (\lambda - \lambda_0)(\mu_0 - \lambda)^{-1}(f, E_0[\mu_0]g_0), $$
Here $G_1(\lambda), G_2(\lambda), G_3(\lambda)$ and $G_4(\lambda)$ are analytic in a neighborhood of $\mu_0$. By the symbol $E_0[\mu_0]$ we denote $E_0(\mu_0 + 0) - E_0(\mu_0 - 0)$.

Using equations (12), (13), (14) and (15), we see that

$$Q_1^s(\lambda) + 1 = G_6(\lambda)(\mu_0 - \lambda)^{-1} + (\lambda - \lambda_0)^2(\lambda - \lambda_0)^{-1}$$

$$\cdot (E_0[\mu_0]g_0, g_0)^2,$$

$$(R_0(\lambda)f, h)[Q(\lambda) + 1] = G_7(\lambda)(\mu_0 - \lambda)^{-1} + (\lambda - \lambda_0)(\lambda - \lambda_0)(\lambda_0 - \lambda)^{-1}$$

$$\cdot (f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h),$$

where $G_6(\lambda), G_7(\lambda), G_8(\lambda)$ are analytic in a neighborhood of $\mu_0$.

Substituting (16), (17), (18), (19) into (6) and (7), we obtain

$$F_1(\lambda) = [G_6(\lambda) - G_7(\lambda)]$$

$$\cdot [G_6(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)^2(\lambda - \lambda_0)^{-1}(E_0[\mu_0]g_0, g_0)^2]^{-1},$$

$$F_2(\lambda) = [G_7(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)(\lambda - \lambda_0)(f, E_0[\mu_0]g_0)(E_0[\mu_0]g_0, h)]$$

$$\cdot [G_7(\lambda)(\mu_0 - \lambda) + (\lambda - \lambda_0)^2(\lambda - \lambda_0)^{-1}(E_0[\mu_0]g_0, g_0)^2]^{-1}.$$
\[(\frac{1}{2}E(\beta) + E(\beta + 0) - \frac{1}{2}E(\alpha) + E(\alpha + 0))f, h)\]

\[= (2\pi i)^{-1} \lim_{\gamma \to +0} \int_\beta^\alpha [F_1(\sigma + i\gamma) - F_1(\sigma - i\gamma)]d\sigma\]

\[+ (2\pi i)^{-1} \lim_{\gamma \to +0} \int_\alpha^\beta \{F_2(\sigma + i\gamma)[\theta(\sigma + i\gamma) Q_1(\sigma + i\gamma) - 1]
\]

\[\cdot [\theta(\sigma + i\gamma) + Q_1(\sigma + i\gamma)]^{-1}
\]

\[- F_2(\sigma - i\gamma)[\theta(\sigma - i\gamma) Q_1(\sigma - i\gamma) - 1]
\]

\[\cdot [\theta(\sigma - i\gamma) + Q_1(\sigma - i\gamma)]^{-1}\}\, d\sigma.\]

The first limit is zero by the analyticity of \(F_1(\lambda)\). Since \(\theta(\lambda) = \theta(\lambda)^{-1}\), \(Q_1(\lambda) = [Q_1(\lambda)]^{-1}\), and \(IQ_1(\lambda) > 0\) for \(I\lambda > 0\) (see [3]), we may put \(\Psi(\lambda) = F_2(\lambda)\) and \(\Phi(\lambda) = \theta(\lambda) Q_1(\lambda) - 1\) in Lemma 1 in order to evaluate the second limit. Hence,

\[\left(\frac{1}{2}E^+(\beta) + E^+(\beta + 0) - \frac{1}{2}E^+(\alpha) + E^+(\alpha + 0)\right)f, h)\]

\[= (\left(\frac{1}{2}E^+(\beta) + E^+(\beta + 0) - \frac{1}{2}E(\alpha) + E(\alpha + 0)\right)f, h)\]

(21)

\[= \int_\alpha^\beta (f, g(\sigma))(g(\sigma), h)[Q_2(\sigma) + 1]^{-1}d\rho(\sigma),\]

where \(\rho(\sigma) = (1/\pi) \lim_{\gamma \to +0} \int_\sigma I\Phi(\mu + i\gamma)d\mu\), and \(E^+(\mu)\) is the spectral function of \(A^+\).

It is understood that the integrand is defined by continuity at the eigenvalues of \(A_0\). From (21) follows equation (4).

Since \(A^+\) is a minimal selfadjoint extension, the Hilbert space \(H^+\) in which \(A^+\) acts is the closed linear hull of the set \(Z\) of elements of the form \([E^+(\beta) - E^+(\alpha)]f\) for arbitrary \(\alpha, \beta\) and arbitrary \(f\) in \(H\). (See Nalmark [7].)

Let \(\{\mu_k\}\) be the eigenvalues of \(A_0\) arranged in order of growth. If \(|k|\) is even, let \(Q_2(\sigma) = [Q_2(\sigma) + 1]^{1/2}\) for \(\mu_k < \sigma < \mu_{k+1}\); if \(|k|\) is odd, let \(Q_2(\sigma) = -[Q_2(\sigma) + 1]^{1/2}\) for \(\mu_k < \sigma < \mu_{k+1}\). If the eigenvalues are bounded above or below, we modify the above definition accordingly for \(\sigma\) greater than the largest eigenvalue or for \(\sigma\) less than the smallest eigenvalue. The definition of \(Q_2(\sigma)\) has been contrived so that

\((f, g(\sigma))[Q_2(\sigma)]^{-1}\) is a continuous function of \(\sigma\).

We define an operator \(V\) on \(Z\) into \(L^2_\nu\) as follows: \(V[E^+(\beta) - E^+(\alpha)]f = \chi_{[\alpha, \beta]}(\sigma)(f, g(\sigma))[Q_2(\sigma)]^{-1}\), where \(\chi_{[\alpha, \beta]}(\sigma)\) is the characteristic function of \([\alpha, \beta]\). From (4) we see that \(||V[E^+(\beta) - E^+(\alpha)]f|| = ||[E^+(\beta) - E^+(\alpha)]f||. We now extend \(V\) linearly to the linear hull of \(Z\) and by continuity to all of \(H^+\). \(V\) is then an isometry of \(H^+\) into \(L^2_\nu\). \(V\) is in fact onto, because the set \(VZ\) is dense in \(L^2_\nu\). To see this, sup-
pose that $k(\sigma) \in L^2_\rho$ and that $k(\sigma)$ is perpendicular to $\chi_{[a, \beta]}(\sigma) (f, g(\sigma)) \cdot [Q_2(\sigma)]^{-1}$ for all intervals $[a, \beta]$ and for all $f$ in $H$. As in [3] it follows that $k(\sigma) = 0$ almost everywhere with respect to $\rho$ for points in the resolvent set of $\mathcal{A}_0$. If $\mu_0$ is an eigenvalue of $\mathcal{A}_0$, we can show that

$$\int_{[\mu_0]} (f, g(\sigma)) [Q_2(\sigma)]^{-1} k(\sigma) d\rho(\sigma) = 0,$$

i.e.,

$$(f, g(\mu_0)) [Q_2(\mu_0)]^{-1} k(\mu_0) d\rho(\mu_0) = 0,$$

where $\Delta \rho(\mu_0)$ is the $\rho$-measure of $\mu_0$. Taking $f = g_0$ in this last equation and using the fact that $(g_0, g(\mu_0)) [Q_2(\mu_0)]^{-1} \neq 0$, we see that $k(\mu_0) = 0$ if $\Delta \rho(\mu_0) \neq 0$. (Note that by $(f, g(\mu_0)) [Q_2(\mu_0)]^{-1}$ we mean the value of the function $(f, g(\sigma)) [Q_2(\sigma)]^{-1}$ defined by continuity at $\sigma = \mu_0$.) Hence, $k(\sigma) = 0$ almost everywhere with respect to $\rho$. This shows that $VZ$ is dense in $L^2_\rho$.

It is not difficult to check that $V$ carries the spectral function of $\mathcal{A}^+$ into the spectral function of the multiplication operator in $L^2_\rho$. This proves the theorem.

**Remark 1.** Suppose $\mathcal{A}^+$ is a selfadjoint extension of $\mathcal{A}$ in $H$ (different from $\mathcal{A}_0$), so that the function $\theta(\lambda)$ in equation (3) is identically equal to a real constant $\theta_0$. Then from equation (2) for $\rho(\sigma)$ and the fact that $\mathcal{A}^+$ is unitarily equivalent to the multiplication operator in $L^2_\rho$, we easily deduce the well-known results that the spectrum of $\mathcal{A}^+$ consists of eigenvalues with no finite limit point and that the eigenvalues of $\mathcal{A}^+$ are all different from those of $\mathcal{A}_0$. From this it follows that every point is of regular type for $\mathcal{A}$.

**Remark 2.** Let $F(\lambda) = \left[\theta(\lambda) - i\lambda \theta_0\right]\left[\theta(\lambda) + i\lambda \theta_0\right]^{-1}$ for $\lambda > 0$. Then $F(\lambda)$ is analytic in the upper half-plane and $|F(\lambda)| \leq 1$. Suppose that $F(\lambda)$ is continuous down to the real axis and that $|F(\lambda)| < 1$ on the real axis. Then one may check that $\Phi(\lambda)$ is continuous (i.e., may be defined continuously) down to the real axis, and $\mathcal{I} \Phi(\lambda)$ is positive on the real axis. Hence, $\rho(\mu) = (1/\pi) \int_0^\infty \mathcal{I} \Phi(\mu) d\mu$ is absolutely continuous, and the multiplication operator in $L^2_\rho$ is unitarily equivalent to the multiplication operator in $L^2$. Thus, if $F(\lambda)$ is continuous down to the real axis and $|F(\lambda)| < 1$ on the real axis, it follows that $\mathcal{A}^+$ is unitarily equivalent to the multiplication operator in $L^2$.

**Remark 3.** If $\mathcal{A}$ is a singular Sturm-Liouville operator, one may use the work of Kac [4, Theorem 7] in order to consider the case that there exists a selfadjoint extension $\mathcal{A}_0$ with a singular spectral function. Let $\mathcal{A}$ be the symmetric operator in $L^2[0, \infty)$ generated by the differential operator $l[y] = -d^2 y/dx^2 + q(x)y$ where $q$ is real and con-
tinuous for \(0 \leq x < \infty\). Suppose that the differential operator is in the limit point case and that the spectral function \(\rho_0(\sigma)\) corresponding to the Weyl function \(m(\lambda)\) is singular (i.e., \(A\) has a selfadjoint extension \(A_0\) with a singular spectral function). According to Straus [9] there is a one-one correspondence between the minimal selfadjoint dilations \(A^+\) of \(A\) and the class of functions \(\theta(\lambda)\) which are analytic in the upper half-plane and have nonnegative imaginary part. If \(E(\lambda)\) is the spectral function of \(A\) corresponding to the minimal selfadjoint extension \(A^+\), then by the work of Kac, for \(f \in L^2(0, \infty)\),

\[
\left[ \frac{1}{2} \{ E(\beta) + E(\alpha + 0) \} - \frac{1}{2} \{ E(\alpha) + E(\alpha + 0) \} \right] f(x) = \int_\alpha^\beta \int_0^\infty f(y) u(y, \sigma) dy u(x, \sigma) d\rho(\sigma)
\]

where

\[
\rho(\sigma) = \frac{1}{\pi} \lim_{r \to 0} \int_0^r i \Phi(\mu + i\tau) d\mu,
\]

\[
\Phi(\lambda) = [\theta(\lambda) m(\lambda) - 1] [\theta(\lambda) + m(\lambda)]^{-1},
\]

and \(u(x, \sigma)\) is a solution of \(i [y] - \sigma y = 0, y(0) + m(\sigma)y'(0) = 0\). The latter equation is to be interpreted as \(y'(0) = 0\) in the event that \(\lim_{\lambda \to 0} m(\lambda)\) does not exist as a finite number.

REFERENCES


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